

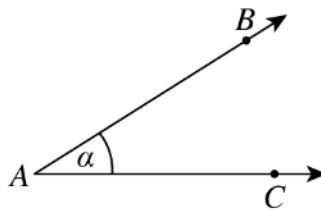
## Reference: Geometry Key Concepts



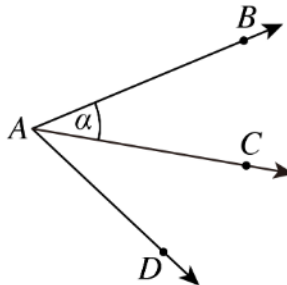
### Concept 1: Special Pairs of Angles

#### 1. Notation of Angles:

The angle below can be denoted by 'angle  $BAC$ ' and written as  $\angle BAC$ , or it can be denoted by the angle 'alpha' and labelled  $\alpha$ .



Alternatively, an angle may be denoted by the label on the vertex. For instance, the angle  $\alpha$  above may be denoted by  $\angle A$ , which is read as 'angle  $A$ '. This notation can be ambiguous if there is more than one angle at the vertex, as shown in the figure below.



#### 2. Special pairs of Angles:

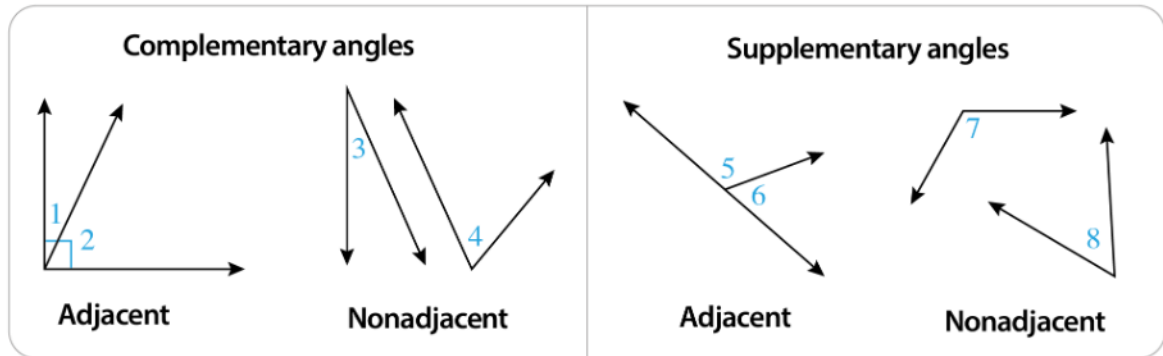
**Complementary:** Complementary angles are two angles that the sum of whose measures is  $90^\circ$ .

**Supplementary:** Supplementary angles are two angles that sum of whose measures is  $180^\circ$ .

**3. Adjacent Angles:** Adjacent angles are two angles that share a common vertex and side but have no common interior points.

**4. Vertical Angles:** If the sides of two angles form two pairs of opposite rays, they are vertical angles.

**5. Linear Pairs:** A linear pair is a **pair of adjacent angles formed when two lines intersect**. The sum of their measures is  $180^\circ$ .



**Theorem 1-1:** Vertical angles are congruent.

### Concept 2: Perpendicular Lines

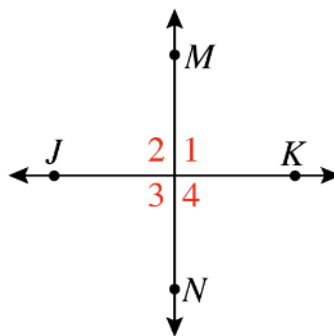
Perpendicular lines are two lines that intersect to form right angles.

Theorems about perpendicular lines:

Theorem 1-2: If two lines are perpendicular, then they form congruent adjacent angles.

Theorem 1-3: If two lines form congruent adjacent angles, then they are perpendicular.

Theorem 1-4: If the exterior sides of two adjacent acute angles are perpendicular, then the angles are complementary.



### Concept 3: Planning a Proof

A **proof of the theorem** includes following five parts:

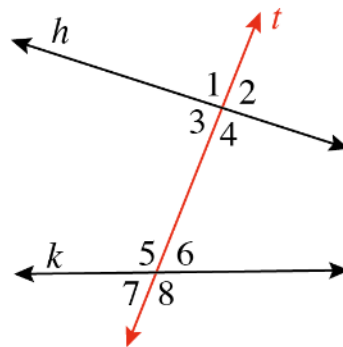
1. Statement of the theorem
2. A diagram that represents the given information
3. A list of what is given
4. A list of what you are to prove
5. A series of statements and reasons that lead from the given information to the statement that is to be proven



#### Concept 4: Definition of Parallel Lines and Planes and Transversal

**Parallel lines:** coplanar lines that do not intersect.

A **transversal** is a line that intersects two or more coplanar lines in different points. In the diagram below,  $t$  is a transversal of  $h$  and  $k$ . The angles formed have special names.



**Corresponding angles:** A pair of angles which lie on the same side of the transversal.

Example:  $\angle 2$  and  $\angle 6$ .

**Alternate interior angles:** A pair of angles which lie on the different sides of the transversal.

Example:  $\angle 3$  and  $\angle 6$ .

**Same-side interior angles:** A pair of angles which lie on the same side of the transversal and lie between lines  $a$  and  $b$ .

Example:  $\angle 3$  and  $\angle 5$ .



#### Concept 5: Theorem of Parallel Lines

**Postulate 2-1:** If two parallel lines are cut by a transversal, then corresponding angles are congruent.

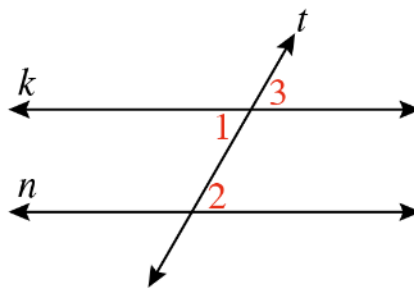
We can use this postulate to prove the following three theorems easily.

**Theorem 2-1:**

If two parallel lines are cut by a transversal, then alternate interior angles are congruent.

Given:  $k \parallel n$ ; transversal  $t$  cuts  $k$  and  $n$ .

Prove:  $\angle 1 \cong \angle 2$

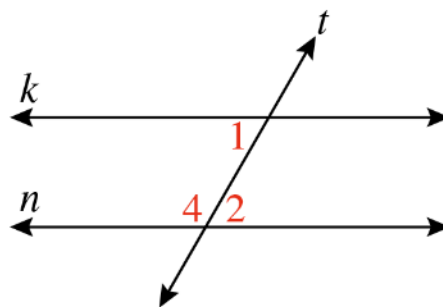


**Theorem 2-2:**

If two parallel lines are cut by a transversal, then same-side interior angles are supplementary.

Given:  $k \parallel n$ ; transversal  $t$  cuts  $k$  and  $n$ .

Prove:  $\angle 1$  is supplementary to  $\angle 4$ .



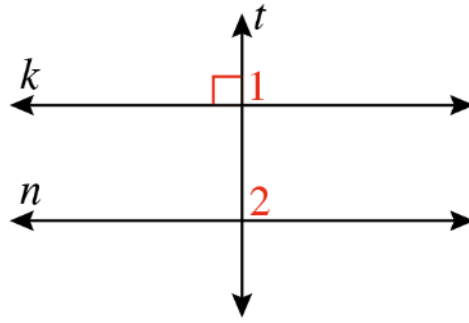
**Theorem 2-3:**

If a transversal is perpendicular to one of two parallel lines, then it is also perpendicular to the other one.

Given: Transversal  $t$  cuts  $k$  and  $n$ ;

$t \perp k$ ;  $k \parallel n$

Prove:  $t \perp n$



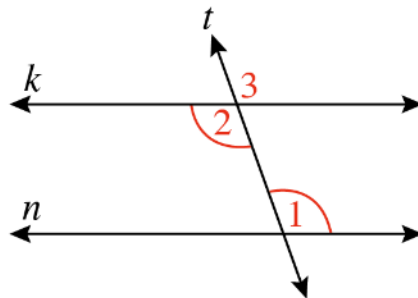
### Concept 6: Proving Lines Parallel

**Postulate 2-2:** If two lines are cut by a transversal and corresponding angles are congruent, then the lines are parallel.

The next three theorems can be proved by using postulate 2-2.

#### **Theorem 2-4:**

If two lines are cut by a transversal and alternate interior angles are congruent, then the two lines are parallel.



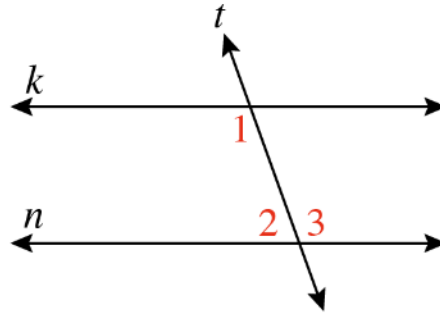
Given: Transversal  $t$  cuts lines  $k$  and  $n$ ;

$$\angle 1 \cong \angle 2$$

Prove:  $k \parallel n$

#### **Theorem 2-5:**

If two lines are cut by a transversal and same-side interior angles are supplementary, then the two lines are parallel.

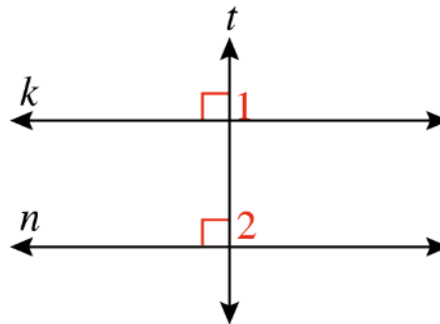


Given: Transversal  $t$  cuts lines  $k$  and  $n$ ;

$\angle 1$  is supplementary to  $\angle 2$

Prove:  $k \parallel n$

**Theorem 2-6:**



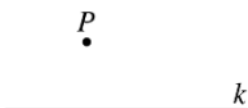
In a plane, two lines perpendicular to the same line are parallel.

Given:  $k \perp t$ ;  $n \perp t$

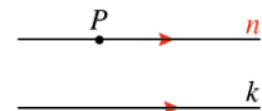
Prove:  $k \parallel n$

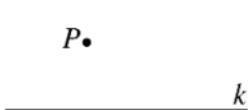
**Theorem 2-7:** Through a point outside a line, there is exactly one line parallel to the given line.

**Theorem 2-8:** Through a point outside a line, there is exactly one line perpendicular to the given line.

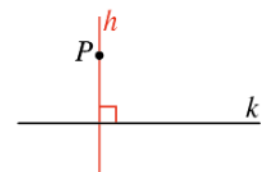
Given this:  $P$   


Theorem 2-7 says that line  $n$  exists and is unique.



Given this:  $P$   


Theorem 2-8 says that line  $h$  exists and is unique.

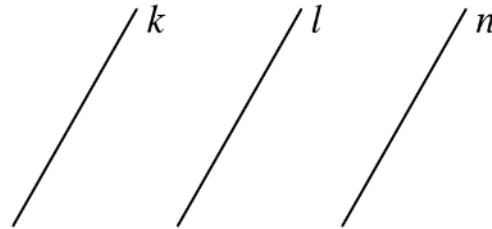


**Theorem 2-9:**

Two lines parallel to a third line are parallel to each other.

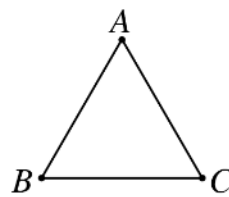
Given:  $k \parallel l; k \parallel n$

Prove:  $l \parallel n$



### Concept 7: Interior and Exterior Angles of Triangles

1. A **triangle** is formed by joining three noncollinear points. Each triangle has three vertices, three sides and three angles. Triangle  $ABC$  is shown below.

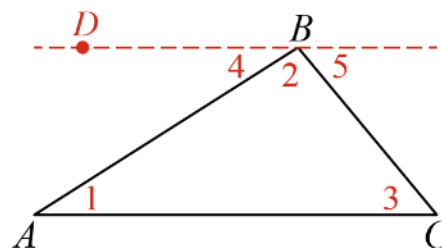


Vertices of  $\triangle ABC$ : points  $A, B, C$

Sides of  $\triangle ABC$ :  $\overline{AB}, \overline{BC}, \overline{CA}$

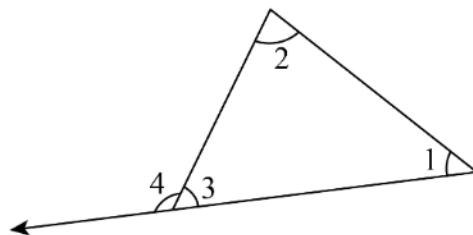
Angles of  $\triangle ABC$ :  $\angle A, \angle B, \angle C$

2. **Interior angles of the triangle:** Angles which are formed by two sides of the triangle.



Example:  $\angle 1, \angle 2, \angle 3$

**Theorem 3-1:** The sum of measures of the interior angles of a triangle is  $180^\circ$ .



**3. Exterior angles of the triangle:** Angles which are formed by one side of the triangle and the extension of another side.

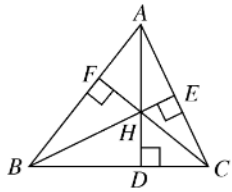
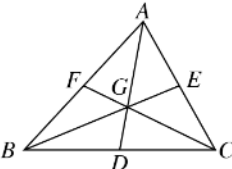
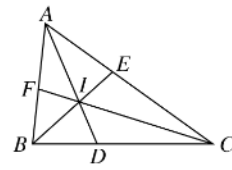
Example:  $\angle 4$

**Remote interior angles:** Interior angles which are not adjacent to the exterior angle.

Example:  $\angle 1$  and  $\angle 2$  are remote angles to  $\angle 4$ .

**Theorem 3-2:** The measure of an exterior angle of a triangle equals the sum of the measures of the two remote interior angles.

#### 4. Altitude, Median, and Angle Bisector

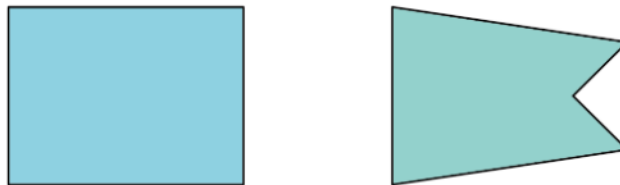
|                | Definition                                                                                                                        | Example                                                                               |
|----------------|-----------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------|
| Altitude       | The altitude of a triangle is <b>perpendicular to the opposite side</b> . Thus, it forms 90 degrees angle with the opposite side. |  |
| Median         | A median of a triangle is <b>a line segment joining a vertex to the midpoint of the opposite side, thus bisecting that side</b> . |  |
| Angle Bisector | The angle bisector of an angle of a triangle is <b>a straight line that divides the angle into two congruent angles</b> .         |  |

## Concept 8: Angles of Polygons

A **polygon** can be defined as a flat or plane, two-dimensional closed shape with straight sides.

A **convex polygon** is a simple polygon (not self-intersecting) in which no line segment between two points on the boundary ever goes outside the polygon.

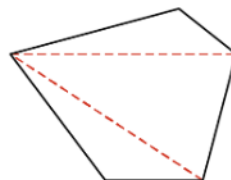
Otherwise, the polygon will be a **concave polygon**.



Listed below are some special names for polygons:

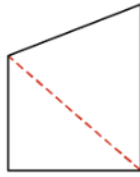
| <i>Number of Sides</i> | <i>Name</i>   |
|------------------------|---------------|
| 3                      | triangle      |
| 4                      | quadrilateral |
| 5                      | pentagon      |
| 6                      | hexagon       |
| 8                      | octagon       |
| 10                     | decagon       |
| $n$                    | $n$ -gon      |

A segment joining two nonconsecutive vertices is a diagonal of the polygon. The diagonals of the pentagon at bottom are indicated by dashes.

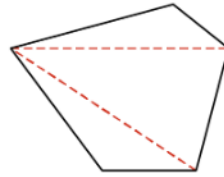


One can use the diagonals to divide a polygon into several parts to get the following theorem:

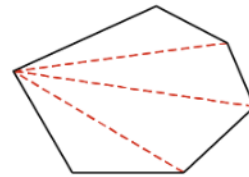
**Theorem 3-3:** The sum of the measures of the angles of a convex polygon with  $n$  sides is  $(n - 2)180^\circ$ .



4 sides, 2 triangles  
Angle sum =  $2(180)$



5 sides, 3 triangles  
Angle sum =  $3(180)$



6 sides, 4 triangles  
Angle sum =  $4(180)$

Similarly, the sum of the measures of the angles of a concave polygon with  $n$  sides is also  $(n - 2)180^\circ$ .

From theorem 3-3, the theorem 3-4 can be deduced:

**Theorem 3-4:** The sum of the measures of the exterior angles of any convex polygon, one angle at each vertex, is  $360^\circ$ .

This can be easily proved since the sum of measures of the exterior angles and interior angles is  $(180n)^\circ$ .

A regular polygon is a polygon with congruent sides and equal angles.



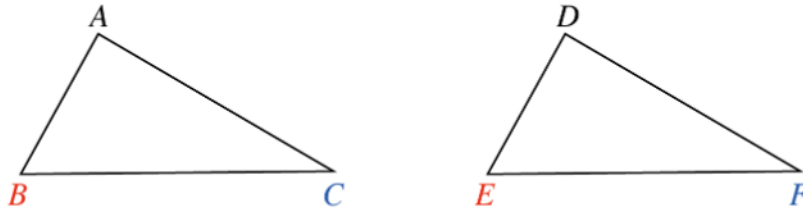
### Concept 9: Definition and Properties of Congruent Figures

**Congruent Figures:** Whenever two figures have the **same size and shape**, they are called **congruent figures**.

**Congruent Triangles:** Whenever two **triangles** have the **same size and shape**, they are called **congruent triangles**. **If we get a new figure by translating, rotating or reflecting a known figure, then the new figure will be congruent to the previous figure.**

Congruent parts of figures are sides or angles that have the same relative position.

As shown in the figure below, triangle  $ABC$  is congruent to triangle  $DEF$ , we can denote it as  $\triangle ABC \cong \triangle DEF$ .



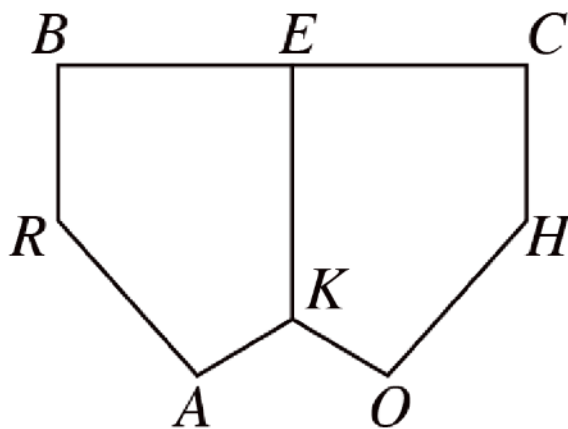
Corresponding Points:  $A \leftrightarrow D, B \leftrightarrow E, C \leftrightarrow F$

Corresponding Angles:  $\angle A \leftrightarrow \angle D, \angle B \leftrightarrow \angle E, \angle \_\_\_\_ \leftrightarrow \angle \_\_\_\_$

Corresponding Sides:  $\overline{AB} \leftrightarrow \overline{DE}, \overline{BC} \leftrightarrow \overline{EF}, \_\_\_\_ \leftrightarrow \_\_\_\_ ,$

### Congruent polygons:

Two polygons are congruent if and only if their vertices can be matched up so that the corresponding parts (sides and angles) of the polygons are congruent.



$$BRAKE \cong CHOKE$$

### Notation of congruent polygons:

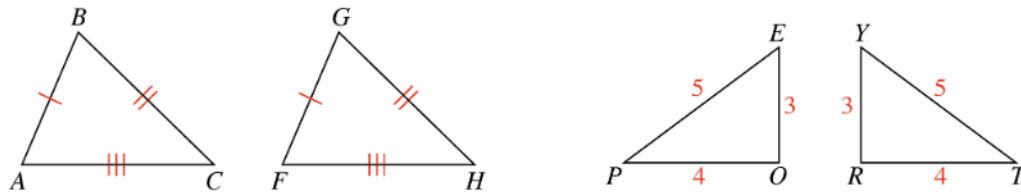
$$BRAKE \cong CHOKE$$

$BRAKE$  and  $CHOKE$  are congruent.

Properties of Congruent Figures: If two figures are congruent, then the corresponding angles are congruent and the corresponding sides are congruent.

### Concept 10: Some Ways to Prove Triangles Congruent

(1) SSS Postulate: If **three sides** of one triangle are congruent to **three sides** of another triangle, then the triangles are **congruent**.



By the SSS Postulate,  $\triangle ABC \cong \triangle FGH$ , and  $\triangle POE \cong \triangle TRY$ .

**Complete process of proof:**

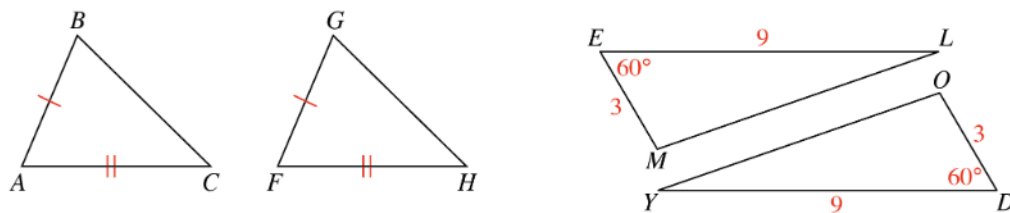
We have  $AB = FG$ ,  $BC = GH$ ,  $CA = HF$ .

Thus, by the **SSS Postulate**,  $\triangle ABC \cong \triangle FGH$ .

We have  $PO = \underline{\hspace{1cm}}$ ,  $OE = \underline{\hspace{1cm}}$ ,  $\underline{\hspace{1cm}} = TY$ .

Thus, by the **SSS Postulate**,  $\triangle POE \cong \triangle TRY$ .

(2) SAS Postulate: If **two sides** and **the included angle** of one triangle are congruent to **two sides** and **the included angle** of another triangle, then the triangles are **congruent**.



**Complete process of proof:**

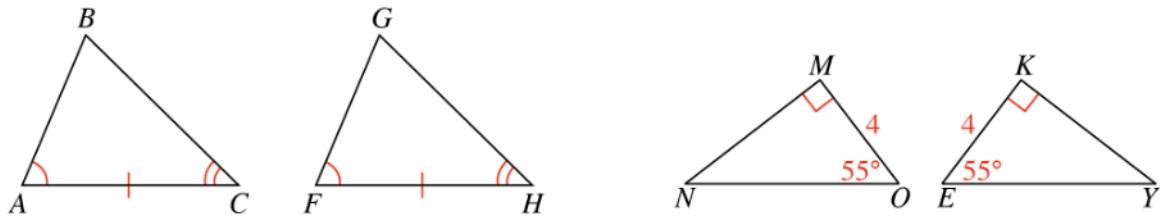
We have  $AB = FG$ ,  $AC = FH$ ,  $m\angle A = m\angle F$ .

Thus, by the **SAS Postulate**,  $\triangle ABC \cong \triangle FGH$ .

We have  $\underline{\hspace{1cm}} = \underline{\hspace{1cm}}$ ,  $\underline{\hspace{1cm}} = \underline{\hspace{1cm}}$ ,  $m\angle \underline{\hspace{1cm}} = m\angle \underline{\hspace{1cm}}$ .

Thus, by the **SAS Postulate**,  $\triangle MEL \cong \triangle ODY$ .

(3) ASA Postulate: If **two angles** and the **included side** of one triangle are congruent to **two angles** and the **included side** of another triangle, then the triangles are **congruent**.



**Complete process of proof:**

We have  $m\angle A = m\angle F$ ,  $AC = FH$ ,  $m\angle C = m\angle H$ .

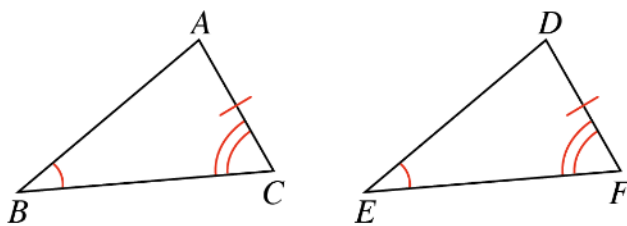
Thus, by the **ASA Postulate**,  $\triangle ABC \cong \triangle FGH$ .

We have  $m\angle O = m\angle$  \_\_,  $OM =$  \_\_,  $m\angle M = m\angle$  \_\_.

Thus, by the **ASA Postulate**,  $\triangle MON \cong \triangle KEY$ .

 **Concept 11: Other Ways of Proving Triangles Congruent**

**(1) AAS Theorem (Based on ASA Postulate):** If **two angles** and a **non-included side** of one triangle are congruent to the corresponding parts of another triangle, then the triangles are **congruent**.



**Complete process of proof:**

We have  $AC = DF$ ,  $m\angle B = m\angle E$ ,  $m\angle C = m\angle F$ .

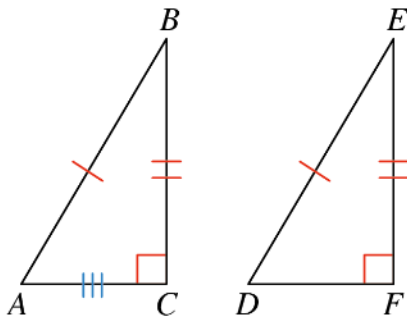
Because  $m\angle A = 180^\circ - m\angle B - m\angle C$ ;

$m\angle D = 180^\circ - m\angle E - m\angle F$ .

Thus  $m\angle A = m\angle D$ .

Thus, by the **ASA Postulate**,  $\triangle ABC \cong \triangle DEF$ .

(2) HL Theorem: If the **hypotenuse** and a **leg** of one **right triangle** are congruent to the corresponding parts of another **right triangle**, then the triangles are **congruent**.



**Process of proof:**

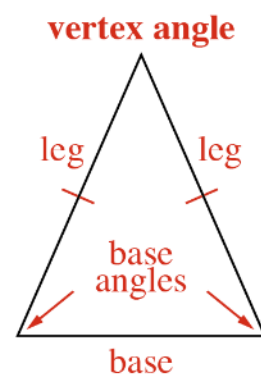
We have  $AB = DE$ ,  $BC = EF$ ,  $m\angle C = m\angle F = 90^\circ$ .

Thus, by the **HL Theorem**,  $\triangle ABC \cong \triangle DEF$ .



### Concept 12: Definition of Isosceles and Equilateral Triangles

1. **Isosceles Triangle**: A triangle with two congruent sides. These congruent sides are called **legs**, and the third side is called the **base**. The two equal angles at the base are called **base angles** and the angle opposite the base is called **vertex angle**.



2. An **equilateral triangle** is a triangle with all three sides of **equal length**. Since the three sides are equal, the three angles, opposite to the equal sides, are equal in measure. Therefore, it is also called an **equiangular triangle**, where each angle measures 60 degrees.



### Concept 13: Properties of Isosceles Triangles

**The Isosceles Triangle Theorem**: If two **sides** of a triangle are congruent, then the **angles** opposite those sides are congruent.

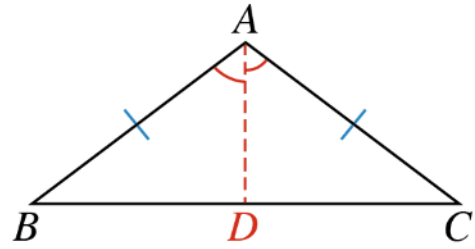
**Proof:**

Draw the bisector of  $\angle A$ , so we have  $\angle BAD \cong \angle CAD$ .

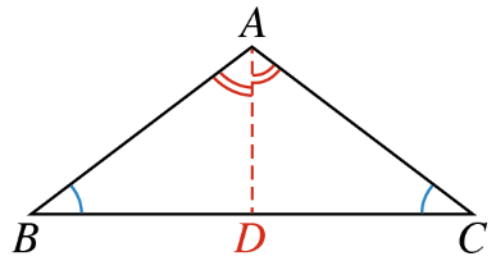
Given:  $AB = AC$ ,  $AD = AD$ , we have  $\triangle BAD \cong \triangle CAD$

by **SAS** Postulate.

So corresponding angles  $\angle B \cong \angle C$ .



**The Converse of Isosceles Triangle Theorem:** If two **angles** of a triangle are congruent, then the **sides** opposite those **angles** are congruent.



**Proof:**

Draw the bisector of  $\angle A$ , so we have  $\angle BAD \cong \angle CAD$ .

Given that  $\angle B \cong \angle C$ ,  $AD = AD$ , we have  $\triangle BAD \cong \triangle CAD$  by **AAS** Theorem.

So corresponding sides  $AB = AC$ .

Important properties of **isosceles triangle**:

For any isosceles triangle, the following three line segments coincide:

- (1) the altitude from the vertex perpendicular to the base.
- (2) the angle bisector from the vertex to the base.
- (3) the median from the vertex to the midpoint of the base.

If two of these three lines are coincide in a triangle, then this triangle is an isosceles triangle.



**Concept 14: Using Auxiliary Lines**

**Purpose of adding auxiliary lines:** highlight necessary information to make the steps toward the proof more recognizable.

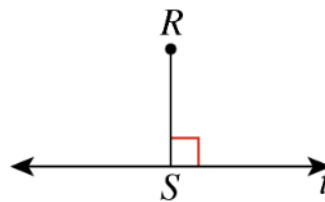
**Basic methods** in adding auxiliary lines:

- ① Connect points  $AB$ ;
- ② Extend line  $AB$  to the point  $C$  to make  $AB=BC$ ;
- ③ Extend line  $AB$  to intersect with the extension of line  $CD$  at point  $E$ ;
- ④ On line  $AB$ , intercept  $AB=BC$ , and connect  $AC$ ;
- ⑤ Construct the parallel line of line  $CD$  passing point  $A$  and intercept with line  $BE$  at point  $F$ ;
- ⑥ Construct the perpendicular line of line  $CD$  on point  $A$  with the foot point  $H$ .

### Concept 15: Angle Bisectors

**Distance from a point to a line (or plane):**

The distance from a point to a line (or plane) is defined to be the length of the perpendicular segment from the point to the line (or plane). As shown in the figure below,  $\overline{RS} \perp t$  at  $S$ .  $RS$  is the distance from  $R$  to line  $t$ .



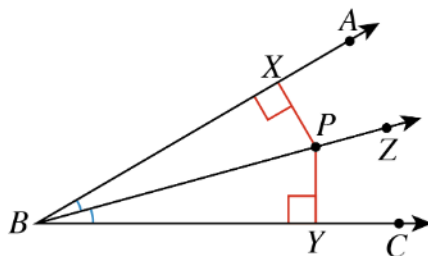
**Definition of Angle Bisector**

An angle bisector of an angle is a line (or ray or segment) that **divides the angle into two equal angles**. As shown in the figure below, segment  $BP$  is the angle bisector of  $\angle B$ .

**Angle Bisector Theorem**

If a point lies on the bisector of an angle, then the point is equidistant from the sides of the angle.

In the figure below,  $\overrightarrow{BP}$  bisects  $\angle B$ ;  $\overline{PX} \perp \overline{BA}$  at  $X$ ;  $\overline{PY} \perp \overline{BC}$  at  $Y$ . We have  $PX = PY$ .

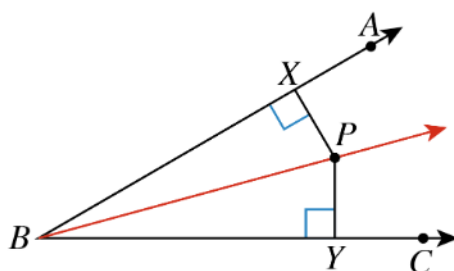


### Converse of Angle Bisector Theorem

If a point is **equidistant** from the sides of an angle, then **the point lies on the bisector** of the angle.

In the figure below,  $\overline{PX} \perp \overline{BA}$  at  $X$ ,  $\overline{PY} \perp \overline{BC}$  at  $Y$ . We have  $PX = PY$ .

Then  $\overrightarrow{BP}$  bisects  $\angle B$ .



### Concept 16: Indirect Proof

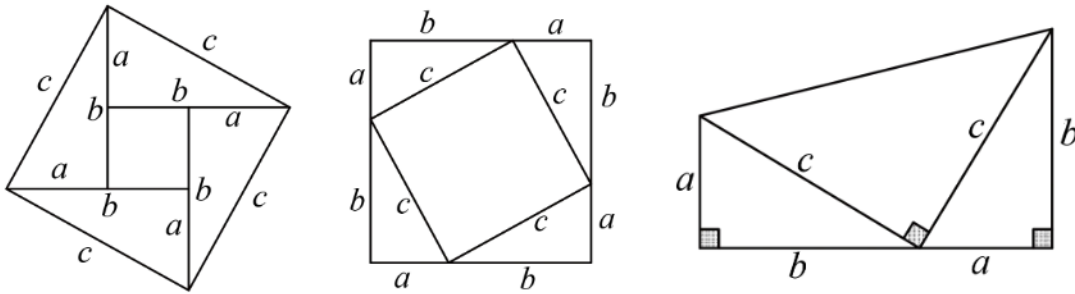
Indirect proof: instead of showing that the conclusion to be proved is true, you can show that all of the alternatives are false.

Steps to write an indirect proof:

1. Assume temporarily that the statement to be proved is not true. (Find the negation of the statement to be proved)
2. Reason logically to find a contradiction of a known fact.
3. Point out that the temporary assumption must be false and the statement to be proved must then be true.

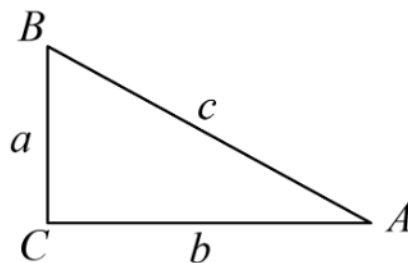
## Concept 17: The Pythagorean Theorem

(1) Pythagorean Theorem: In a right triangle, the sum of the squares of the two legs is equal to the square of the hypotenuse.



The figures above illustrate some classical proofs of the Pythagorean Theorem.

(2) Converse of Pythagorean Theorem: In a triangle, if the square of one side is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.



(3) A Pythagorean triple is a set of three nonzero whole numbers that satisfy the Pythagorean Theorem.

Pythagorean triples that are often used:

$(3, 4, 5)$ ,  $(6, 8, 10)$ ,  $(5, 12, 13)$ ,  $(7, 24, 25)$

If  $(a, b, c)$  is a Pythagorean triple, then for arbitrary positive integer  $n$ ,  $(na, nb, nc)$  is a Pythagorean triple.

(4) In a right triangle, the length of the median drawn from the vertex of the right angle equals half the length of the triangle's hypotenuse.

In a triangle, if the length of the median drawn from a vertex to a side equals half the length of the this side, then it is a right triangle and this side is the hypotenuse.

- 1 As shown in the figure, in  $\triangle ABC$ ,  $\overline{CD} \perp \overline{AB}$  at point  $D$ .  $E$  is the midpoint of  $\overline{AC}$ . If  $AD = 6$ , and  $DE = 5$ , then the length of  $\overline{CD}$  is \_\_\_\_\_.

