

2021 AMC 10A Spring

1 What is the value of $(2^2 - 2) - (3^2 - 3) + (4^2 - 4)$ (2021 AMC Spring 10A, Question #1)

- A. 1 B. 2 C. 5 D. 8 E. 12

Answer D

Solution Solution 1:

$(4 - 2) - (9 - 3) + (16 - 4) = 2 - 6 + 12 = 8$. This corresponds to (D) 8 .

Solution 2:

$$\begin{aligned} (2^2 - 2) - (3^2 - 3) + (4^2 - 4) &= 2(2 - 1) - 3(3 - 1) + 4(4 - 1) \\ &= 2(1) - 3(2) + 4(3) \\ \text{We have} \qquad \qquad \qquad &= 2 - 6 + 12 \\ &= (D) 8 \end{aligned}$$

2 Portia's high school has 3 times as many students as Lara's high school. The two high schools have a total of 2600 students. How many students does Portia's high school have? (2021 AMC Spring 10A, Question #2)

- A. 600 B. 650 C. 1950 D. 2000 E. 2050

Answer C

Solution Solution 1:

The following system of equations can be formed with P representing the number of students in Portia's high school and L representing the number of students in Lara's high school: $P = 3L$, $P + L = 2600$. Substituting $P = 3L$ gives $4L = 2600$. Solving for L gives $L = 650$. Since we need to find P , we multiply 650 by 3 to get $P = (C) 1950$.

Solution 2:

Suppose Lara's high school has x students, so Portia's high school has $3x$ students.

We have $x + 3x = 2600$, or $4x = 2600$. The answer is $3x = 2600 \cdot \frac{3}{4} = 650 \cdot 3 =$ (C) 1950

- 3 The sum of two natural numbers is 17,402. One of the two numbers is divisible by 10. If the units digit of that number is erased, the other number is obtained. What is the difference of these two numbers?(2021 AMC Spring 10A, Question #3)

A. 10272 B. 11700 C. 13362 D. 14238 E. 15426

Answer D

Solution Solution 1:

The units digit of a multiple of 10 will always be 0. We add a 0 whenever we multiply by 10. So, removing the units digit is equal to dividing by 10.

Let the smaller number (the one we get after removing the units digit) be a . This means the bigger number would be $10a$. We know the sum is $10a + a = 11a$ so

$11a = 17402$. So $a = 1582$. The difference is $10a - a = 9a$. So, the answer is $9(1582) =$ (D) 14,238.

Solution 2:

Since the ones place of a multiple of 10 is 0, this implies the other integer has to end with a 2 since both integers sum up to a number that ends with a 2. Thus, the ones place of the difference has to be $10 - 2 = 8$, and the only answer choice that ends with an 8 is (D) 14,238. Another quick solution is to realize that the sum is represents a number n added to $10n$. The difference is $9n$, which is $\frac{9}{11}$ of the given sum.

- 4 A cart rolls down a hill, travelling 5 inches the first second and accelerating so that during each successive 1-second time interval, it travels 7 inches more than during the previous 1 second interval. The cart takes 30 seconds to reach the bottom of the hill. How far, in inches, does it travel?(2021 AMC Spring 10A, Question #4)

A. 215

B. 360

C. 2992

D. 3195

E. 3242

Answer D

Solution Solution 1:

Since Distance = Speed $\cdot$ Time, we seek the sum

$$5 \cdot 1 + 12 \cdot 1 + 19 \cdot 1 + 26 \cdot 1 + \cdots = 5 + 12 + 19 + 26 + \cdots \text{ in which there are 30 terms.}$$

The last term is $5 + 7 \cdot (30 - 1) = 208$. Therefore, the requested sum is

$$5 + 12 + 19 + 26 + \cdots + 208 = \frac{5 + 208}{2} \cdot 30 = \text{(D) } 3195$$

Recall that to find the sum of an arithmetic series, we take the average of the first and last terms, then multiply by the number of terms: $\text{Sum} = \frac{\text{First} + \text{Last}}{2} \cdot \text{Count}.$

Solution 2:

The distance (in inches) traveled within each 1 -second interval is:

$5, 5 + 1(7), 5 + 2(7), \dots, 5 + 29(7)$. This is an arithmetic sequence so the total distance travelled, found by summing them up is:

$$\text{number of terms} \cdot \text{average of terms} = \text{number of terms} \cdot \frac{\text{first term} + \text{last term}}{2}. \text{ Or,}$$

$$30 \cdot \frac{5 + 5 + 29(7)}{2} = 15 \cdot 213 = \text{(D) } 3195$$

- 5 The quiz scores of a class with $k > 12$ students have a mean of 8 . The mean of a collection of 12 of these quiz scores is 14 . What is the mean of the remaining quiz scores in terms of k ?(2021 AMC Spring 10A, Question #5)

A. $\frac{14 - 8}{k - 12}$

B. $\frac{8k - 168}{k - 12}$

C. $\frac{14}{12} - \frac{8}{k}$

D. $\frac{14(k - 12)}{k^2}$

E. $\frac{14(k - 12)}{8k}$

Answer B

Solution Solution 1:

The total score of the class is $8k$, and the total score of the 12 quizzes is $12 \cdot 14 = 168$.

Therefore, for the remaining quizzes ($k - 12$ of them), the total score is $8k - 168$. Their mean score is (B) $\frac{8k - 168}{k - 12}$.

Solution 2:

Set $k = 13$. The answer is the same as the last student's quiz score, which is

$8 \cdot 13 - 14 \cdot 12 < 0$. From the answer choices, only (B) $\frac{8k - 168}{k - 12}$ is negative at $k = 13$.

- 6 Chantal and Jean start hiking from a trailhead toward a fire tower. Jean is wearing a heavy backpack and walks slower. Chantal starts walking at 4 miles per hour. Halfway to the tower, the trail becomes really steep, and Chantal slows down to 2 miles per hour. After reaching the tower, she immediately turns around and descends the steep part of the trail at 3 miles per hour. She meets Jean at the halfway point. What was Jean's average speed, in miles per hour, until they meet?(2021 AMC Spring 10A, Question #6)

- A. $\frac{12}{13}$ B. 1 C. $\frac{13}{12}$ D. $\frac{24}{13}$
E. 2

Answer A

Solution Solution 1:

Let $2d$ miles be the distance from the trailhead to the fire tower, where $d > 0$. When

Chantal meets Jean, the two have traveled for

$\frac{d}{4} + \frac{d}{2} + \frac{d}{3} = d \left(\frac{1}{4} + \frac{1}{2} + \frac{1}{3} \right) = d \left(\frac{3}{12} + \frac{6}{12} + \frac{4}{12} \right) = \frac{13}{12}d$ hours. At that point, Jean has traveled for d miles, so his average speed is $\frac{d}{\frac{13}{12}d} =$ (A) $\frac{12}{13}$ miles per hour.

Solution 2:

We will follow the same template as shown in Solution 1, except that we will replace d with a convenient constant.

Let 24 miles be the distance from the trailhead to the fire tower. When Chantal meets Jean, the two have traveled for $\frac{12}{4} + \frac{12}{2} + \frac{12}{3} = 3 + 6 + 4 = 13$ per hour.

- 7 Tom has a collection of 13 snakes, 4 of which are purple and 5 of which are happy. He observes that - all of his happy snakes can add, - none of his purple snakes can subtract, and - all of his snakes that can't subtract also can't add. Which of these conclusions can be drawn about Tom's snakes?(2021 AMC Spring 10A, Question #7)
- A. Purple snakes can add
B. Purple snakes are happy.
C. Purple snakes are happy.
D. Happy snakes are not purple
E. Happy snakes can't subtract

Answer D

Solution Solution 1:

We are given that

happy	$\implies$	can add,
purple	$\implies$	cannot subtract,
cannot subtract	$\implies$	cannot add.

Two solutions follow from here:

Solution 1.1 (Intuitive) Combining (2) and (3) gives happy $\implies$ can add. Clearly, the answer is (D). Recall that every conditional statement $p \implies q$ is always logically equivalent to its contrapositive $\neg q \implies \neg p$.

Solution 1.2 (Rigorous) Combining (1), (2) and (3) gives Applying the hypothetical syllogism to (**), we conclude that purple $\implies$ not happy, whose contrapositive is happy $\implies$ not purple. Therefore, the answer is (D).

Remark The conclusions in the other choices do not follow from (**): (A) purple $\implies$ can add (B) purple $\implies$ happy (C) can add $\implies$ purple (E) happy $\implies$ cannot subtract

Solution 2:

From Solution 1.1, we can also see this through the process of elimination.

Statement *A* is false because purple snakes cannot add. *B* is false as well because since happy snakes can add and purple snakes can not add, purple snakes are not

happy snakes. E is false using the same reasoning, purple snakes are not happy snakes so happy snakes can subtract since purple snakes cannot subtract. C is false since snakes that can add are happy, not purple. That leaves statement D.

- 8 When a student multiplied the number 66 by the repeating decimal, $1.\underline{abab}\dots = 1.\underline{ab}$, where a and b are digits, he did not notice the notation and just multiplied 66 times $1.\underline{ab}$. Later he found that his answer is 0.5 less than the correct answer. What is the 2 -digit number $\underline{ab}$?(2021 AMC Spring 10A, Question #8)

A. 15 B. 30 C. 45 D. 60 E. 75

Answer E

Solution Solution 1:

We are given that $66(1.\underline{ab}) - 0.5 = 66(1.\underline{ab})$, from which

$$\begin{aligned} 66(1.\underline{ab}) - 66(1.\underline{ab}) &= 0.5 \\ 66(\underline{1}.\underline{ab} - \underline{1}.\underline{ab}) &= 0.5 \\ 66(\underline{0}.\underline{00ab}) &= 0.5 \\ 66\left(\frac{1}{100} \cdot \underline{0}.\underline{ab}\right) &= \frac{1}{2} \\ \underline{0}.\underline{ab} &= \frac{25}{33} \\ \underline{0}.\underline{ab} &= 0.\overline{75} \\ \underline{ab} &= (\mathbf{E})75. \end{aligned}$$

Solution 2:

It is known that $\underline{0}.\underline{ab} = \frac{a}{99}$ and $\underline{0}.\underline{ab} = \frac{ab}{100}$. Let $x = \underline{ab}$. We have $66\left(1 + \frac{x}{99}\right) - 66\left(1 + \frac{x}{100}\right) = 0.5$ Expanding and simplifying give $\frac{x}{150} = 0.5$, so $x = (\mathbf{E})75$.

- 9 What is the least possible value of $(xy - 1)^2 + (x + y)^2$ for real numbers x and y ?(2021 AMC Spring 10A, Question #9)

A. 0 B. $\frac{1}{4}$ C. $\frac{1}{2}$ D. 1 E. 2

Answer D

Solution Solution 1:

Expanding, we get that the expression is $x^2 + 2xy + y^2 + x^2y^2 - 2xy + 1$ or $x^2 + y^2 + x^2y^2 + 1$. By the Trivial Inequality (all squares are nonnegative) the minimum value for this is (D) 1, which can be achieved at $x = y = 0$.

Solution 2:

We expand the original expression, then factor the result by grouping:

$$\begin{aligned}(xy - 1)^2 + (x + y)^2 &= (x^2y^2 - 2xy + 1) + (x^2 + 2xy + y^2) \\ &= x^2y^2 + x^2 + y^2 + 1 \\ &= x^2(y^2 + 1) + (y^2 + 1) \\ &= (x^2 + 1)(y^2 + 1).\end{aligned}$$

Clearly, both factors are

positive. By the Trivial Inequality, we have $(x^2 + 1)(y^2 + 1) \geq (0 + 1)(0 + 1) = \text{(D)}1$

10 Which of the following is equivalent to

$$(2 + 3)(2^2 + 3^2)(2^4 + 3^4)(2^8 + 3^8)(2^{16} + 3^{16})(2^{32} + 3^{32})(2^{64} + 3^{64})? \text{(2021 AMC Spring 10A,}$$

Question #10)

- A. $3^{127} + 2^{127}$
- B. $3^{127} + 2^{127} + 2 \cdot 3^{63} + 3 \cdot 2^{63}$
- C. $3^{128} - 2^{128}$
- D. $3^{128} + 2^{128}$
- E. 5^{127}

Answer C

Solution Solution 1:

By multiplying the entire equation by $3 - 2 = 1$, all the terms will simplify by difference of squares, and the final answer is (C) $3^{128} - 2^{128}$. Additionally, we could also multiply the entire equation (we can let it be equal to x) by $2 - 3 = -1$. The terms again simplify by difference of squares. This time, we get

$-x = 2^{128} - 3^{128} \Rightarrow x = 3^{128} - 2^{128}$. Both solutions yield the same answer.

Solution 2:

If you weren't able to come up with the $(3 - 2)$ insight, then you could just notice that the answer is divisible by $2 + 3 = 5$, and $2^2 + 3^2 = 13$. We can then use Fermat's Little Theorem for $p = 5, 13$ on the answer choices to determine which of the answer choices are divisible by both 5 and 13. This is (C) $3^{128} - 2^{128}$

11 For which of the following integers b is the base- b number $2021_b - 221_b$ not divisible by 3
(2021 AMC Spring 10A, Question #11)

- A. 3 B. 4 C. 6 D. 7 E. 8

Answer E

Solution Solution 1:

$$\begin{aligned} 2021_b - 221_b &= (2021_b - 21_b) - (221_b - 21_b) \\ &= 2000_b - 200_b \\ &= 2b^3 - 2b^2 \qquad \text{which is divisible by 3 unless} \\ &= 2b^2(b - 1), \end{aligned}$$

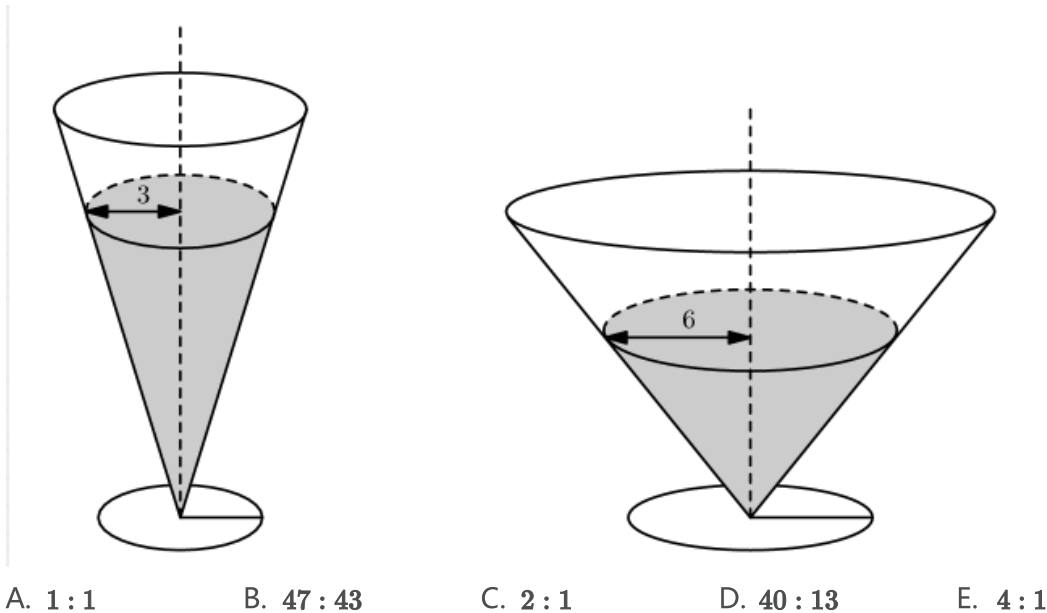
We have

$b \equiv 2 \pmod{3}$. The only choice congruent to 2 modulo 3 is (E) 8.

Solution 2:

Vertically subtracting $2021_b - 221_b$, we see that the ones place becomes 0, and so does the b^1 place. Then, we perform a carry (make sure the carry is in base b). Let $b - 2 = A$. Then, we have our final number as $1A00_b$. Now, when expanding, we see that this number is simply $b^3 - (b - 2)^2$. Now, notice that the final number will only be congruent to $b^3 - (b - 2)^2 \equiv 0 \pmod{3}$. If either $b \equiv 0 \pmod{3}$, or if $b \equiv 1 \pmod{3}$ (because note that $(b - 2)^2$ would become $\equiv 1 \pmod{3}$, and b^3 would become $\equiv 1 \pmod{3}$ as well, and therefore the final expression would become $1 - 1 \equiv 0 \pmod{3}$). Therefore, b must be $\equiv 2 \pmod{3}$. Among the answers, only 8 is $\equiv 2 \pmod{3}$, and therefore our answer is (E) 8.

- 12 Two right circular cones with vertices facing down as shown in the figure below contains the same amount of liquid. The radii of the tops of the liquid surfaces are 3-cm and 6-cm. Into each cone is dropped a spherical marble of radius 1-cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the rise of the liquid level in the wide cone?(2021 AMC Spring 10A, Question #12)



Answer E

Solution Solution 1:

Initial Scenario Let the heights of the narrow cone and the wide cone be h_1 and h_2 , respectively. We have the following table:

	Base Radius	Height	Volume
Narrow Cone	3	h_1	$\frac{1}{3}\pi(3)^2h_1 = 3\pi h_1$
Wide Cone	6	h_2	$\frac{1}{3}\pi(6)^2h_2 = 12\pi h_2$

Equating the volumes gives $3\pi h_1 = 12\pi h_2$, which simplifies to $\frac{h_1}{h_2} = 4$. Furthermore, by similar triangles: - For the

narrow cone, the ratio of the base radius to the height is $\frac{3}{h_1}$, which always remains constant. - For the wide cone, the ratio of the base radius to the height is $\frac{6}{h_2}$, which always remains constant. Two solutions follow from here:

Solution 1.1 (Properties of Fractions)

Final Scenario

For the narrow cone and the wide cone, let their base radii be $3x$ and $6y$ (for some $x, y > 1$), respectively. By the similar triangles discussed above, their heights must be h_1x and h_2y , respectively. We have the following table:

	Base Radius	Height	Volume
Narrow Cone	$3x$	h_1x	$\frac{1}{3}\pi(3x)^2(h_1x) = 3\pi h_1x^3$
Wide Cone	$6y$	h_2y	$\frac{1}{3}\pi(6y)^2(h_2y) = 12\pi h_2y^3$

Equating the volumes gives $3\pi h_1x^3 = 12\pi h_2y^3$, which simplifies to $x^3 = y^3$, or $x = y$.

Finally, the requested ratio is $\frac{h_1x - h_1}{h_2y - h_2} = \frac{h_1(x - 1)}{h_2(y - 1)} = \frac{h_1}{h_2} = 4 : 1$. Remarks 1. This

solution uses the following property of fractions: For unequal positive numbers a, b, c

and d , if $\frac{a}{b} = \frac{c}{d} = k$, then $\frac{a \pm c}{b \pm d} = \frac{bk \pm dk}{b \pm d} = \frac{(b \pm d)k}{b \pm d} = k$. 2. This solution shows that,

regardless of the shape or the volume of the solid dropped into each cone, the

requested ratio stays the same as long as the solid sinks to the bottom and is

completely submerged without spilling any liquid.

Solution 1.2 (Bash)

Final Scenario

For the narrow cone and the wide cone, let their base radii be r_1 and r_2 , respectively;

let their rises of the liquid levels be Δh_1 and Δh_2 , respectively. We have the following

	Base Radius	Height	Volume
table:	r_1	$h_1 + \Delta h_1$	$\frac{1}{3}\pi r_1^2(h_1 + \Delta h_1)$
	r_2	$h_2 + \Delta h_2$	$\frac{1}{3}\pi r_2^2(h_2 + \Delta h_2)$

above, we get $\frac{3}{h_1} = \frac{r_1}{h_1 + \Delta h_1} \Rightarrow r_1 = \frac{3}{h_1}(h_1 + \Delta h_1)$,

$\frac{6}{h_2} = \frac{r_2}{h_2 + \Delta h_2} \Rightarrow r_2 = \frac{6}{h_2}(h_2 + \Delta h_2)$. The volume of the marble dropped into each cone is $\frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$. Now, we set up an equation for the volume of the narrow cone,

$$\frac{1}{3}\pi r_1^2(h_1 + \Delta h_1) = 3\pi h_1 + \frac{4}{3}\pi$$

$$\frac{1}{3}r_1^2(h_1 + \Delta h_1) = 3h_1 + \frac{4}{3}$$

$$\frac{1}{3}\left(\frac{3}{h_1}(h_1 + \Delta h_1)\right)^2(h_1 + \Delta h_1) = 3h_1 + \frac{4}{3}$$

then express Δh_1 in terms of h_1 :

$$\frac{3}{h_1^2}(h_1 + \Delta h_1)^3 = 3h_1 + \frac{4}{3}$$

$$(h_1 + \Delta h_1)^3 = h_1^3 + \frac{4h_1^2}{9}$$

$$\Delta h_1 = \sqrt[3]{h_1^3 + \frac{4h_1^2}{9}} - h_1$$

Next, we set up an equation for the volume of the wide cone, then express Δh_2 in terms of h_2 : $\frac{1}{3}\pi r_2^2 (h_2 + \Delta h_2) = 12\pi h_2 + \frac{4}{3}\pi$

Using a similar process from above, we get $\Delta h_2 = \sqrt[3]{h_2^3 + \frac{h_2^2}{9}} - h_2$. Recall that $\frac{h_1}{h_2} = 4$.

$$\begin{aligned}\frac{\Delta h_1}{\Delta h_2} &= \frac{\sqrt[3]{h_1^3 + \frac{4h_1^2}{9}} - h_1}{\sqrt[3]{h_2^3 + \frac{h_2^2}{9}} - h_2} \\ &= \frac{\sqrt[3]{(4h_2)^3 + \frac{4(4h_2)^2}{9}} - 4h_2}{\sqrt[3]{h_2^3 + \frac{h_2^2}{9}} - h_2} \\ &= \frac{\sqrt[3]{4^3 \left(h_2^3 + \frac{h_2^2}{9}\right)} - 4h_2}{\sqrt[3]{h_2^3 + \frac{h_2^2}{9}} - h_2} \\ &= \frac{4\sqrt[3]{h_2^3 + \frac{h_2^2}{9}} - 4h_2}{\sqrt[3]{h_2^3 + \frac{h_2^2}{9}} - h_2} \\ &= (\mathbf{E}) 4 : 1\end{aligned}$$

Therefore, the requested ratio is

Solution 2:

The heights of the cones are not given, so suppose the heights are very large (i.e. tending towards infinity) in order to approximate the cones as cylinders with base radii 3 and 6 and infinitely large height. Then the base area of the wide cylinder is 4 times that of the narrow cylinder. Since we are dropping a ball of the same volume into each cylinder, the water level in the narrow cone/cylinder should rise (E) 4 times as much.

13 What is the volume of tetrahedron $ABCD$ with edge lengths $AB = 2$, $AC = 3$, $AD = 4$, $BC = \sqrt{13}$, $BD = 2\sqrt{5}$, and $CD = 5$?(2021 AMC Spring 10A, Question #13)

- A. 3 B. $2\sqrt{3}$ C. 4 D. $3\sqrt{3}$ E. 6

Answer C

Solution Solution 1:

Drawing the tetrahedron out and testing side lengths, we realize that the $\triangle ACD$, $\triangle ABC$, and $\triangle ABD$ are right triangles by the Converse of the Pythagorean Theorem. It is now easy to calculate the volume of the tetrahedron using the formula for the volume of a pyramid. If we take $\triangle ADC$ as the base, then $\overline{AB}$ must be the altitude. The volume of tetrahedron $ABCD$ is $\frac{1}{3} \cdot \frac{3 \cdot 4}{2} \cdot 2 = (\text{C}) 4$.

Solution 2:

We will place tetrahedron $ABCD$ in the xyz -plane. By the Converse of the Pythagorean Theorem, we know that $\triangle ACD$ is a right triangle. Without the loss of generality, let $A = (0, 0, 0)$, $C = (3, 0, 0)$, $D = (0, 4, 0)$, and $B = (x, y, z)$. We apply the

$$\begin{aligned} \text{Distance Formula to } \overline{BA}, \overline{BC}, \text{ and } \overline{BD}, \text{ respectively: } & x^2 + y^2 + z^2 = 2^2, \\ & (x-3)^2 + y^2 + z^2 = \sqrt{13}^2, \\ & x^2 + (y-4)^2 + z^2 = (2\sqrt{5})^2. \end{aligned}$$

Subtracting (1) from (2) gives $-6x + 9 = 9$, from which $x = 0$. Subtracting (1) from (3) gives $-8y + 16 = 16$, from which $y = 0$. Substituting $(x, y) = (0, 0)$ into (1) produces $z^2 = 4$, or $|z| = 2$. Let the brackets denote areas. Finally, we find the volume of

$$\begin{aligned} V_{ABCD} &= \frac{1}{3} \cdot [ACD] \cdot h_B \\ \text{tetrahedron } ABCD \text{ using } \triangle ACD \text{ as the base: } &= \frac{1}{3} \cdot \left(\frac{1}{2} \cdot AC \cdot AD \right) \cdot |z| \\ &= (\text{C}) 4 \end{aligned}$$

14 All the roots of the polynomial $z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$ are positive integers, possibly repeated. What is the value of B ? (2021 AMC Spring 10A, Question #14)

- A. -88 B. -80 C. -64 D. -41 E. -40

Answer A

Solution Solution1 :

By Vieta's formulas, the sum of the six roots is 10 and the product of the six roots is 16. By inspection, we see the roots are 1, 1, 2, 2, 2, and 2, so the function is $(z-1)^2(z-2)^4 = (z^2 - 2z + 1)(z^4 - 8z^3 + 24z^2 - 32z + 16)$. Therefore, calculating just

the z^3 terms, we get $B = -32 - 48 - 8 = (A) - 88$.

Solution 2 :

Using the same method as Solution 1, we find that the roots are **2, 2, 2, 2, 1**, and 1 .

Note that B is the negation of the 3rd symmetric sum of the roots. Using casework

on the number of 1 's in each of the $\binom{6}{3} = 20$ products $r_a \cdot r_b \cdot r_c$, we obtain

$$B = - \left(\binom{4}{3} \binom{2}{0} \cdot 2^3 + \binom{4}{2} \binom{2}{1} \cdot 2^2 \cdot 1 + \binom{4}{1} \binom{2}{2} \cdot 2 \right) = -(32 + 48 + 8) = (\mathbf{A}) - 88$$

- 15** Values for A, B, C , and D are to be selected from $\{1, 2, 3, 4, 5, 6\}$ without replacement (i.e. no two letters have the same value). How many ways are there to make such choices so that the two curves $y = Ax^2 + B$ and $y = Cx^2 + D$ intersect? (The order in which the curves are listed does not matter; for example, the choices $A = 3, B = 2, C = 4, D = 1$ is considered the same as the choices $A = 4, B = 1, C = 3, D = 2$.) (2021 AMC Spring 10A, Question #15)

A. 30 B. 60 C. 90 D. 180 E. 360

Answer C

Solution Solution1 :

Visualizing the two curves, we realize they are both parabolas with the same axis of symmetry. Now assume that the first equation is above the second, since order doesn't matter. Then $C > A$ and $B > D$. Therefore the number of ways to choose the four integers is $\binom{6}{2} \binom{4}{2} = 90$, and the answer is (C) 90 .

Solution 2:

Setting $y = Ax^2 + B = Cx^2 + D$, we find that $Ax^2 - Cx^2 = x^2(A - C) = D - B$, so $x^2 = \frac{D - B}{A - C} \geq 0$ by the trivial inequality. This implies that $D - B$ and $A - C$ must both be positive or negative. If two distinct values are chosen for (A, C) and (B, D) respectively, there are 2 ways to order them so that both the numerator and

denominator are positive/negative (increasing and decreasing). We must divide by 2 at the end, however, since the 2 curves aren't considered distinct. Calculating, we get $\frac{1}{2} \cdot \binom{6}{2} \binom{4}{2} \cdot 2 = \text{(C)}90$

16 In the following list of numbers, the integer n appears n times in the list for $1 \leq n \leq 200$.

1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ..., 200, 200, ..., 200 What is the median of the numbers in this list?

(2021 AMC Spring 10A, Question #16)

- A. 100.5 B. 134 C. 142 D. 150.5 E. 167

Answer C

Solution Solution 1:

There are $1 + 2 + \dots + 199 + 200 = \frac{(200)(201)}{2} = 20100$ numbers in total. Let the median be k . We want to find the median k such that $\frac{k(k+1)}{2} = 20100/2$ or $k(k+1) = 20100$. Note that $\sqrt{20100} \approx 142$. Plugging this value in as k gives $\frac{1}{2}(142)(143) = 10153$. $10153 - 142 < 10050$, so 142 is the 152 nd and 153 rd numbers, and hence, our desired answer. (C) 142 .

Note that we can derive $\sqrt{20100} \approx 142$ through the formula

$\sqrt{n} = \sqrt{a+b} \approx \sqrt{a} + \frac{b}{2\sqrt{a}+1}$, where a is a perfect square less than or equal to n . We set a to 19600 , so $\sqrt{a} = 140$, and $b = 500$. We then have $n \approx 140 + \frac{500}{2(140)+1} \approx 142$.

approximation by ciceronii Note by Fasolinka (use answer choices): Once you know that the answer is in the 140 s range by the approximation, it is highly improbable for the answer to be anything but C.

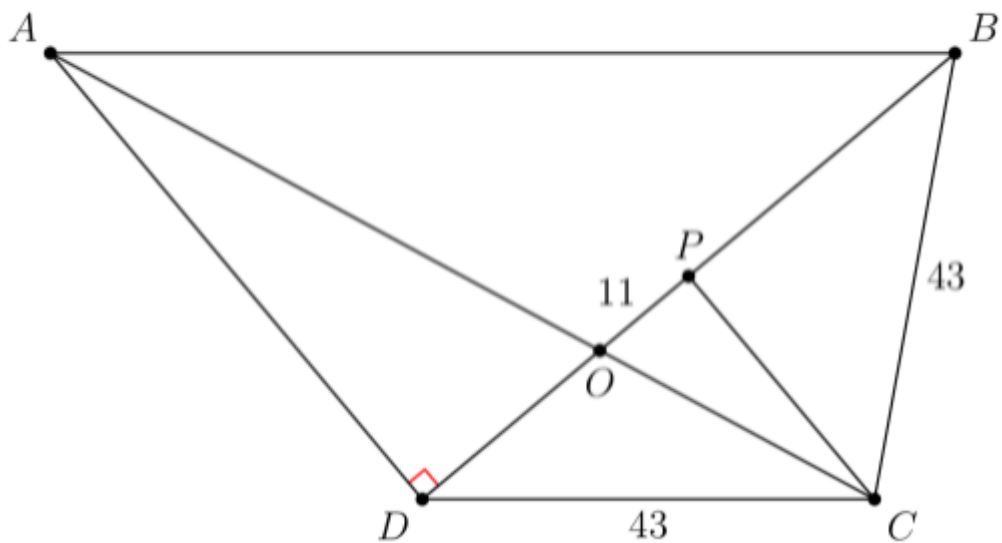
Solution 2:

The x th number of this sequence is $\left\lceil \frac{-1 \pm \sqrt{1+8x}}{2} \right\rceil$ via the quadratic formula. We can see that if we halve x we end up getting $\left\lceil \frac{-1 \pm \sqrt{1+4x}}{2} \right\rceil$. This is approximately the number divided by $\sqrt{2} \cdot \frac{200}{\sqrt{2}} = 141.4$ and since 142 looks like the only number

close to it, it is answer $\square$ (C) 142

- 17 Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$, $BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let O be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let P be the midpoint of $\overline{BD}$. Given that $OP = 11$, the length of AD can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. What is $m + n$? (2021 AMC Spring 10A, Question #17)

Diagram



- A. 65 B. 132 C. 157 D. 194 E. 215

Answer D

Solution1 :

Angle chasing reveals that $\triangle BPC \sim \triangle BDA$, therefore $2 = \frac{BD}{BP} = \frac{AB}{BC} = \frac{AB}{43}$, or

$AB = 86$. Additional angle chasing shows that $\triangle ABO \sim \triangle CDO$, therefore

$2 = \frac{AB}{CD} = \frac{BO}{OD} = \frac{BP + 11}{BP - 11}$, or $BP = 33$ and $BD = 66$. Since $\triangle ADB$ is right, the

Pythagorean theorem implies that $AD = \sqrt{86^2 - 66^2} = 4\sqrt{190}$. The answer is $4 + 190 =$

(D) 194 . - Angle Chasing: If we set $\angle DBC = \alpha$, then we know that $\angle DCB = 180^\circ - 2\alpha$

because $\triangle DBC$ is isosceles. And because $\overline{AB} \parallel \overline{DC}$, we conclude that $\angle ABD = \alpha$ too.

Lastly, because $\triangle BPC$ and $\triangle BDA$ are both right triangles, they are similar by AA.

Solution 2:

Since $\triangle BCD$ is isosceles with base $\overline{BD}$, it follows that median $\overline{CP}$ is also an altitude.

Let $OD = x$ and $CP = h$, so $PB = x + 11$.

Since $\angle AOD = \angle COP$ by vertical angles, we conclude that $\triangle AOD \sim \triangle COP$ by AA,

from which $\frac{AD}{CP} = \frac{OD}{OP}$, or $AD = CP \cdot \frac{OD}{OP} = h \cdot \frac{x}{11}$. Let the brackets denote areas.

Notice that $[AOD] = [BOC]$ (By the same base and height, we deduce that

$[ACD] = [BDC]$. Subtracting $[OCD]$ from both sides gives $[AOD] = [BOC]$.). Doubling

$$\begin{aligned} 2[AOD] &= 2[BOC] \\ OD \cdot AD &= OB \cdot CP \end{aligned}$$

both sides produces $x \left(\frac{hx}{11} \right) = (x + 22)h$ Rearranging and factoring result in

$$x^2 = 11(x + 22).$$

$(x - 22)(x + 11) = 0$, from which $x = 22$. Applying the Pythagorean Theorem to right

$\triangle CPB$, we have $h = \sqrt{43^2 - 33^2} = \sqrt{(43 + 33)(43 - 33)} = \sqrt{760} = 2\sqrt{190}$ Finally, we get

$AD = h \cdot \frac{x}{11} = 4\sqrt{190}$ so the answer is $4 + 190 = \text{(D) } 194$.

18 Let f be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers a and b . Furthermore, suppose that f also has the property that $f(p) = p$ for every prime number p . For which of the following numbers x is $f(x) < 0$? (2021 AMC Spring 10A, Question #18)

A. $\frac{17}{32}$
E. $\frac{25}{11}$

B. $\frac{11}{16}$

C. $\frac{7}{9}$

D. $\frac{7}{6}$

Answer E

Solution Solution 1:

$$f(25) = f\left(\frac{25}{11} \cdot 11\right)$$

From the answer choices, note that $f\left(\frac{25}{11}\right) + f(11)$ On the other hand, we
 $= f\left(\frac{25}{11}\right) + 11.$

$f(25) = f(5 \cdot 5)$
have $= f(5) + f(5)$ Equating the expressions for $f(25)$ produces
 $= 5 + 5$
 $= 10.$

$f\left(\frac{25}{11}\right) + 11 = 10$ from which $f\left(\frac{25}{11}\right) = -1$. Therefore, the answer is (E) $\frac{25}{11}$. Remark

Similarly, we can find the outputs of f at the inputs of the other answer choices: (A)

$f\left(\frac{17}{32}\right) = 7$ (B) $f\left(\frac{11}{16}\right) = 3$ (C) $f\left(\frac{7}{9}\right) = 1$ (D) $f\left(\frac{7}{6}\right) = 2$ Alternatively, refer to

Solutions 2 and 4 for the full processes.

Solution 2:

We know that $f(p) = f(p \cdot 1) = f(p) + f(1)$. By transitive, we have $f(p) = f(p) + f(1)$.

Subtracting $f(p)$ from both sides gives $0 = f(1)$. Also

$$f(2) + f\left(\frac{1}{2}\right) = f(1) = 0 \implies 2 + f\left(\frac{1}{2}\right) = 0 \implies f\left(\frac{1}{2}\right) = -2$$

$$f(3) + f\left(\frac{1}{3}\right) = f(1) = 0 \implies 3 + f\left(\frac{1}{3}\right) = 0 \implies f\left(\frac{1}{3}\right) = -3 \quad \text{In (A) we have}$$

$$f(11) + f\left(\frac{1}{11}\right) = f(1) = 0 \implies 11 + f\left(\frac{1}{11}\right) = 0 \implies f\left(\frac{1}{11}\right) = -11$$

$$f\left(\frac{17}{32}\right) = 17 + 5f\left(\frac{1}{2}\right) = 17 - 5(2) = 7 \quad \text{In (B) we have}$$

$$f\left(\frac{11}{16}\right) = 11 + 4f\left(\frac{1}{2}\right) = 11 - 4(2) = 3. \quad \text{In (C) we have}$$

$$f\left(\frac{7}{9}\right) = 7 + 2f\left(\frac{1}{3}\right) = 7 - 2(3) = 1 \quad \text{In (D) we have}$$

$$f\left(\frac{7}{6}\right) = 7 + f\left(\frac{1}{2}\right) + f\left(\frac{1}{3}\right) = 7 - 2 - 3 = 2 \quad \text{In (E) we have}$$

$$f\left(\frac{25}{11}\right) = 10 + f\left(\frac{1}{11}\right) = 10 - 11 = -1 \quad \text{Thus, our answer is (E) } \frac{25}{11}.$$

19 The area of the region bounded by the graph of $x^2 + y^2 = 3|x - y| + 3|x + y|$ is $m + n\pi$, where m and n are integers. What is $m + n$? (2021 AMC Spring 10A, Question #19)

A. 18

B. 27

C. 36

D. 45

E. 54

Answer E

Solution Solution 1:

In order to attack this problem, we need to consider casework:

Case 1: $|x - y| = x - y, |x + y| = x + y$ Substituting and simplifying, we have

$x^2 - 6x + y^2 = 0$, i.e. $(x - 3)^2 + y^2 = 3^2$, which gives us a circle of radius 3 centered at $(3, 0)$.

Case 2: $|x - y| = y - x, |x + y| = x + y$ Substituting and simplifying again, we have

$x^2 + y^2 - 6y = 0$, i.e. $x^2 + (y - 3)^2 = 3^2$. This gives us a circle of radius 3 centered at $(0, 3)$.

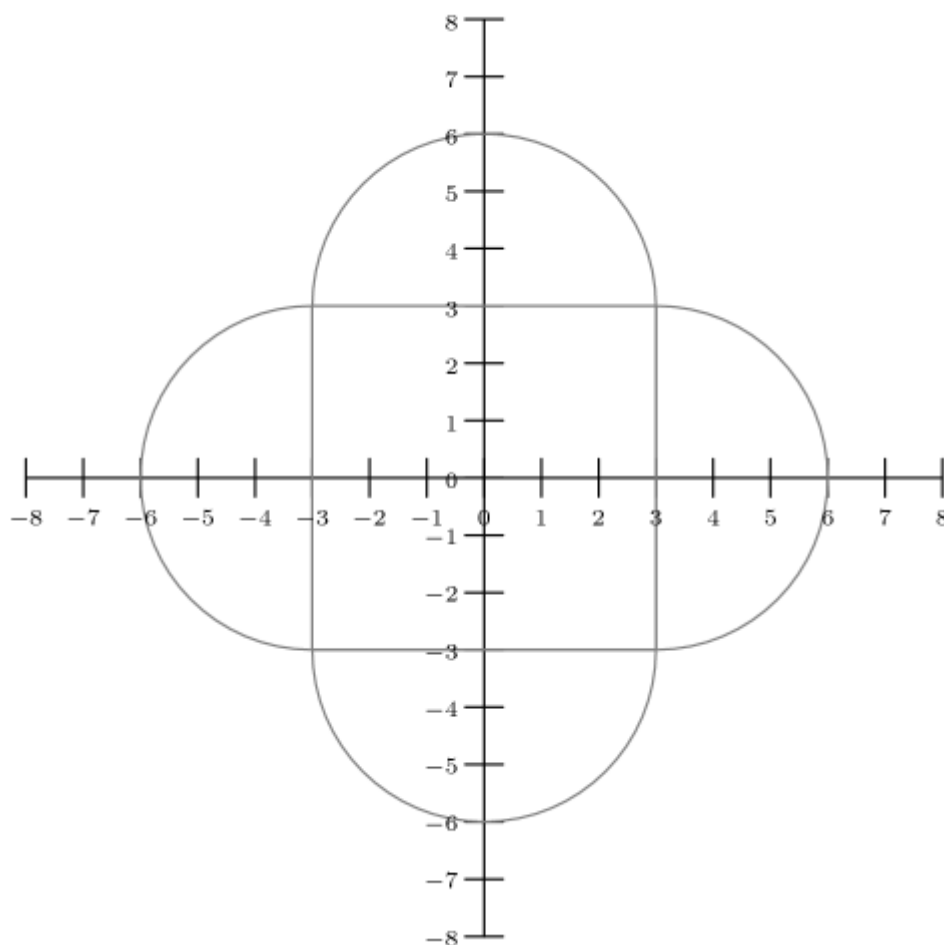
Case 3: $|x - y| = x - y, |x + y| = -x - y$ Doing the same process as before, we have

$x^2 + y^2 + 6y = 0$, i.e. $x^2 + (y + 3)^2 = 3^2$. This gives us a circle of radius 3 centered at $(0, -3)$.

Case 4: $|x - y| = y - x, |x + y| = -x - y$ One last time: we have $x^2 + y^2 + 6x = 0$, i.e.

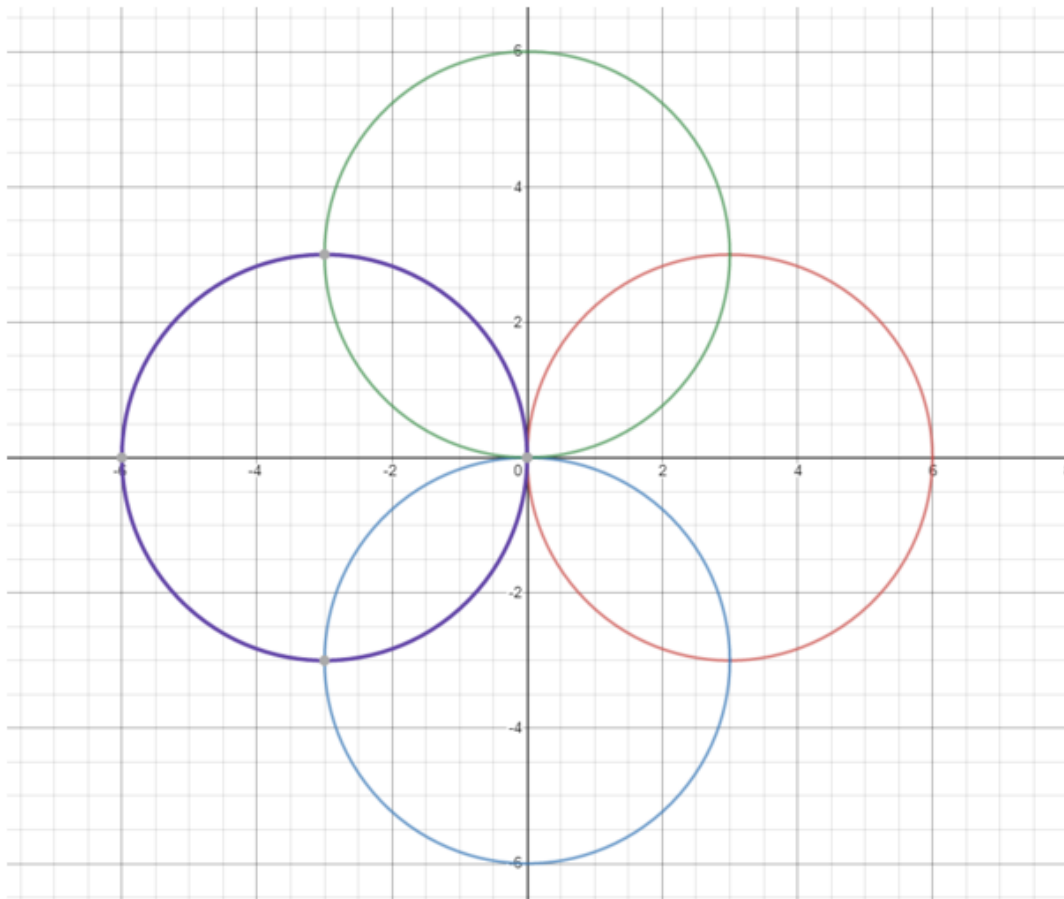
$(x + 3)^2 + y^2 = 3^2$. This gives us a circle of radius 3 centered at $(-3, 0)$. After

combining all the cases and drawing them on the Cartesian Plane, this is what the diagram looks like:



Now, the area of the shaded region is just a square with side length 6 with four semicircles of radius 3. The area is $6 \cdot 6 + 4 \cdot \frac{9\pi}{2} = 36 + 18\pi$. The answer is $36 + 18\pi$ which is (E)54

Solution by Bryguy Remark This problem asks for the area of the union of these four circles:



Solution 2:

Assume $y = 0$. We get that $x = 6$. That means that this figure must contain the points $(0, 6), (6, 0), (0, -6), (-6, 0)$. Now, assume that $x = y$. We get that $x = 3\sqrt{3}$. We get the points $(3, 3), (3, -3), (-3, 3), (-3, -3)$. Since this contains $x^2 + y^2$, assume that there are circles. Therefore, we can guess that there is a center square with area $6 \cdot 6 = 36$ and 4 semicircles with radius 3. We get 4 semicircles with area 4.5π , and therefore the answer is $36 + 18 = (E)54$

20 In how many ways can the sequence 1, 2, 3, 4, 5 be rearranged so that no three consecutive terms are increasing and no three consecutive terms are decreasing? (2021 AMC Spring 10A, Question #20)

- A. 10 B. 18 C. 24 D. 32 E. 44

Answer D

Solution Solution 1:

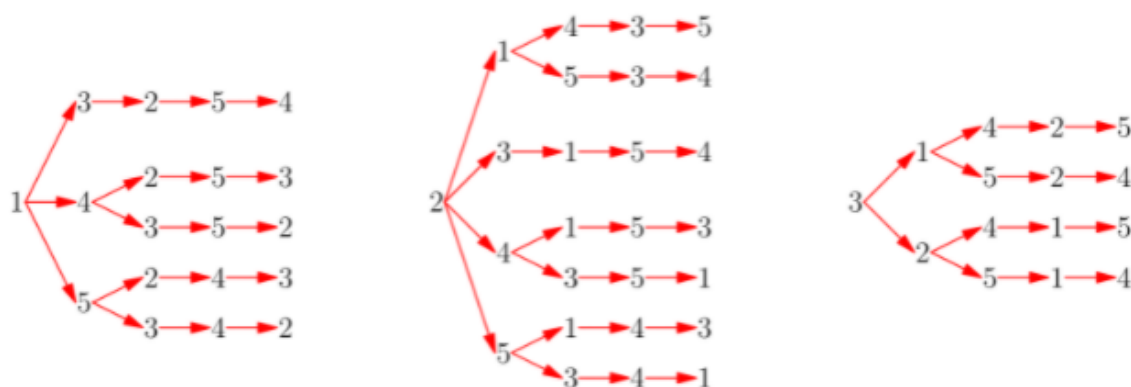
We write out the $5! = 120$ cases, then filter out the valid ones:

13254, 14253, 14352, 15243, 15342, 21435, 21534, 23154, 24153, 24351, 25143, 25341, 31425, 31524, 32415, 32514, 34152, 34251, 35142, 35241, 41325, 41523, 42315, 42513, 43512, 45132, 45231, 51324, 51423, 52314, 52413, 53412.

count these out and get (D) 32 permutations that work.

Solution 2:

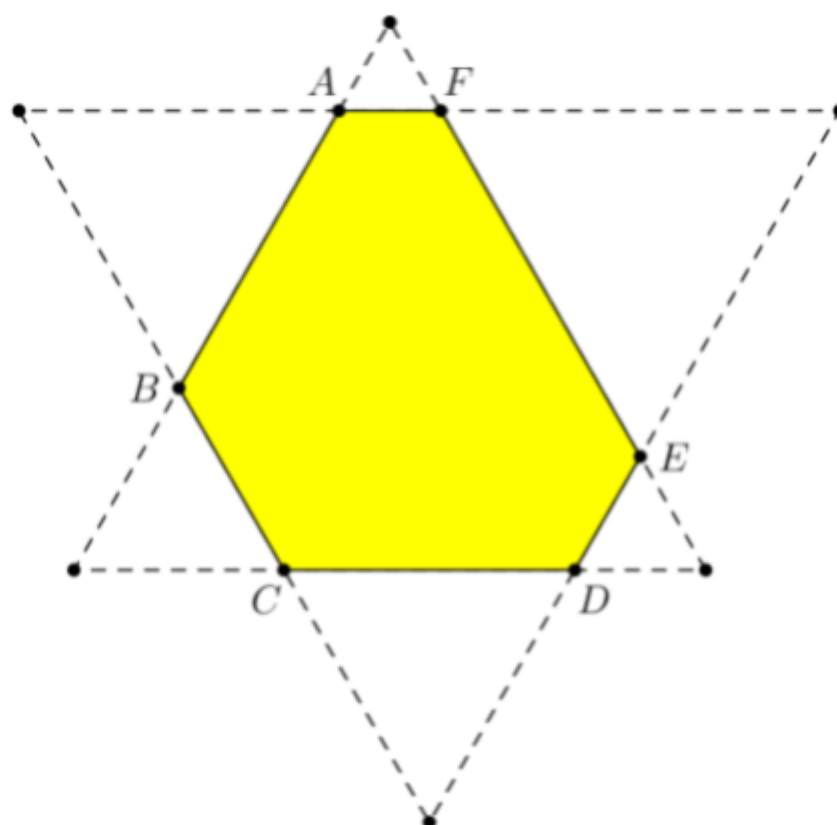
By symmetry with respect to 3, note that $(x_1, x_2, x_3, x_4, x_5)$ is a valid sequence if and only if $(6 - x_1, 6 - x_2, 6 - x_3, 6 - x_4, 6 - x_5)$ is a valid sequence. We enumerate the valid sequences that start with 1, 2, 31, or 32, as shown below:



There are 16 valid sequences that start with 1, 2, 31, or 32. By symmetry, there are 16 valid sequences that start with 5, 4, 35, or 34. So, the answer is $16 + 16 =$ (D) 32.

- 21 Let $ABCDEF$ be an equiangular hexagon. The lines AB , CD , and EF determine a triangle with area $192\sqrt{3}$, and the lines BC , DE , and FA determine a triangle with area $324\sqrt{3}$. The perimeter of hexagon $ABCDEF$ can be expressed as $m + n\sqrt{p}$, where m , n , and p are positive integers and p is not divisible by the square of any prime. What is $m + n + p$? (2021 AMC Spring 10A, Question #21)

Diagram



A. 47

B. 52

C. 55

D. 58

E. 63

Answer C

Solution Let P, Q, R, X, Y , and Z be the intersections

$\overleftrightarrow{AB} \cap \overleftrightarrow{CD}, \overleftrightarrow{CD} \cap \overleftrightarrow{EF}, \overleftrightarrow{EF} \cap \overleftrightarrow{AB}, \overleftrightarrow{BC} \cap \overleftrightarrow{DE}, \overleftrightarrow{DE} \cap \overleftrightarrow{FA}$, and $\overleftrightarrow{FA} \cap \overleftrightarrow{BC}$ respectively. The

sum of the interior angles of any hexagon is 720° . Since hexagon $ABCDEF$ is

equiangular, each of its interior angles is $720^\circ \div 6 = 120^\circ$. By angle chasing, we

conclude that the interior angles of $\triangle PBC, \triangle QDE, \triangle RFA, \triangle XCD, \triangle YEF$, and

$\triangle ZAB$ are all 60° . Therefore, these triangles are all equilateral triangles, from which

$\triangle PQR$ and $\triangle XYZ$ are both equilateral triangles. We are given that

$$[PQR] = \frac{\sqrt{3}}{4} \cdot PQ^2 = 192\sqrt{3} \quad \text{so we get } PQ = 16\sqrt{3} \text{ and } YZ = 36, \text{ respectively. By}$$

$$[XYZ] = \frac{\sqrt{3}}{4} \cdot YZ^2 = 324\sqrt{3}$$

equilateral triangles and segment addition, we find the perimeter of hexagon

$$\begin{aligned}
 AB + BC + CD + DE + EF + FA &= AZ + PC + CD + DQ + YF + FA \\
 ABCDEF : \quad &= (YF + FA + AZ) + (PC + CD + DQ) \\
 &= YZ + PQ \\
 &= 36 + 16\sqrt{3}
 \end{aligned}$$

Finally, the answer is $36 + 16 + 3 = \text{(C)}55$.

22 Hiram's algebra notes are 50 pages long and are printed on 25 sheets of paper; the first sheet contains pages 1 and 2, the second sheet contains pages 3 and 4, and so on. One day he leaves his notes on the table before leaving for lunch, and his roommate decides to borrow some pages from the middle of the notes. When Hiram comes back, he discovers that his roommate has taken a consecutive set of sheets from the notes and that the average (mean) of the page numbers on all remaining sheets is exactly 19. How many sheets were borrowed?(2021 AMC Spring 10A, Question #22)

- A. 10 B. 13 C. 15 D. 17 E. 20

Answer B

Solution Solution 1:

Suppose the roommate took sheets a through b , or equivalently, page numbers $2a - 1$ through $2b$. Because there are $(2b - 2a + 2)$ numbers taken,

$$\begin{aligned}
 \frac{(2a - 1 + 2b)(2b - 2a + 2)}{2} + 19(50 - (2b - 2a + 2)) &= \frac{50 \cdot 51}{2} \implies (2a + 2b - 39)(b - a + 1) \\
 &= \frac{50 \cdot 13}{2} = 25 \cdot 13.
 \end{aligned}$$

The first possible solution that comes to mind is if

$2a + 2b - 39 = 25, b - a + 1 = 13 \implies a + b = 32, b - a = 12$, which indeed works, giving $b = 22$ and $a = 10$. The answer is $22 - 10 + 1 = \text{(B)} 13$

Solution 2:

Suppose the smallest page number borrowed is k , and n pages are borrowed. It follows that the largest page number borrowed is $k + n - 1$. We have the following preconditions: 1. n pages are borrowed means that $\frac{n}{2}$ sheets are borrowed, from which n must be even. 2. k must be odd, as the smallest page number borrowed is

on the right side (odddnumbered). $3 \cdot 1 + 2 + 3 + \dots + 50 = \frac{51(50)}{2} = 1275$. 4. The sum of the page numbers borrowed is $\frac{(2k + n - 1)n}{2}$. Together, we have

$$\begin{aligned} \frac{1275 - \frac{(2k+n-1)n}{2}}{50-n} &= 19 \\ 1275 - \frac{(2k+n-1)n}{2} &= 19(50-n) \\ 2550 - (2k+n-1)n &= 38(50-n) \\ 2550 - (2k+n-1)n &= 1900 - 38n \\ 650 &= (2k+n-39)n. \end{aligned}$$

The factors of 650 are 1, 2, 5, 10, 13, 25, 26, 50, 65, 130, 325, 650. Since n is even, we only have a few cases to consider:

$\vec{n}$	$2k + \vec{k} - 39$	$\vec{k}$
2	325	181
10	65	47
26	25	19
50	13	1
130	5	-43
650	1	-305

Since $1 \leq k \leq 49$, only $k = 47, 19, 1$ are possible: - If $k = 47$, then there will not be sufficient pages when we take 10 pages out starting from page 47. - If $k = 1$, then the average page number of all remaining sheets will be undefined, as there will be no sheets remaining after we take 50 pages (25 sheets) out starting from page 1. Therefore, the only possibility is $k = 19$. We conclude that $n = 26$ pages, or $\frac{n}{2} = (B) 13$ sheets, are borrowed.

- 23 Frieda the frog begins a sequence of hops on a 3×3 grid of squares, moving one square on each hop and choosing at random the direction of each hop-up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she "wraps around" and jumps to the opposite edge. For example if Frieda begins in the center square and makes two hops "up", the first hop would place her in the top row middle square, and the second hop would cause Frieda to jump to the opposite edge, landing in the bottom row middle square. Suppose Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four

hops?(2021 AMC Spring 10A, Question #23)

A. $\frac{9}{16}$
E. $\frac{13}{16}$

B. $\frac{5}{8}$

C. $\frac{3}{4}$

D. $\frac{25}{32}$

Answer D

Solution Solution 1:

We will use complementary counting. First, the frog can go left with probability $\frac{1}{4}$.

We observe symmetry, so our final answer will be multiplied by 4 for the 4 directions, and since $4 \cdot \frac{1}{4} = 1$, we will ignore the leading probability. From the left, she either goes left to another edge $\left(\frac{1}{4}\right)$ or back to the center $\left(\frac{1}{4}\right)$. Time for some casework.

Case 1: She goes back to the center. Now, she can go in any 4 directions, and then has 2 options from that edge. This gives $\frac{1}{2}$. – End case 1

Case 2: She goes to another edge (rightmost). Subcase 1: She goes back to the left edge. She now has 2 places to go, giving $\frac{1}{2}$ Subcase 2: She goes to the center. Now any move works. $\frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot 1 = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$ for this case. –End case 2 She goes back to the center in Case 1 with probability $\frac{1}{4}$, and to the right edge with probability $\frac{1}{4}$

So, our answer is $\frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{3}{8} = \frac{1}{4} \left(\frac{1}{2} + \frac{3}{8} \right) = \frac{1}{4} \cdot \frac{7}{8} = \frac{7}{32}$ But, don't forget complementary counting. So, we get $1 - \frac{7}{32} = \frac{25}{32} \Rightarrow D$.

Solution 2:

We can draw a state diagram with three states: center, edge, and corner. Denote center by M, edge by E , and corner by C . There are a few ways Frieda can reach a corner in four or less moves: EC, EEC, EEEEC, EMEC. Then, calculating the probabilities of each of these cases happening, we have

$$1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} \cdot 1 \cdot \frac{1}{2} = \frac{25}{32}, \text{ so the answer is D}$$

The interior of a quadrilateral is bounded by the graphs of $(x + ay)^2 = 4a^2$ and $(ax - y)^2 = a^2$, where a is a positive real number. What is the area of this region in terms of a , valid for all $a > 0$? (2021 AMC Spring 10A, Question #24)

- A. $\frac{8a^2}{(a+1)^2}$ B. $\frac{4a}{a+1}$
 C. $\frac{8a}{a+1}$ D. $\frac{8a^2}{a^2+1}$
 E. $\frac{8a}{a^2+1}$

Answer D

Solution Solution 1 :

The cases for $(x + ay)^2 = 4a^2$ are $x + ay = \pm 2a$, or two parallel lines. We rearrange each case and construct the table below:

Case	Line's Equation	$\vec{x}$ -intercept	$\vec{y}$ -intercept	Slope
1	$x + ay - 2a = 0$	$2a$	2	$-\frac{1}{a}$
2	$x + ay + 2a = 0$	$-2a$	-2	$-\frac{1}{a}$

The cases for $(ax - y)^2 = a^2$ are $ax - y = \pm a$, or two parallel lines. We rearrange each case and construct the table below:

Case	Line's Equation	$\vec{x}$ -intercept	$\vec{y}$ -intercept	Slope
1*	$ax - y - a = 0$	1	$-a$	a
2*	$ax - y + a = 0$	-1	a	a

Since the slopes of intersecting lines (1) $\cap$ (1*), (1) $\cap$ (2*), (2) $\cap$ (1*), and (2) $\cap$ (2*) are negative reciprocals, we get four right angles, from which the quadrilateral is a rectangle. Two solutions follow from here:

Solution 1.1 (Distance Between Parallel Lines) Recall that for constants A, B, C_1 and C_2 , the distance d between parallel lines $\begin{cases} Ax + By + C_1 = 0 \\ Ax + By + C_2 = 0 \end{cases}$ is $d = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}$. From this formula: - The distance between lines (1) and (2) is $\frac{4a}{\sqrt{1 + a^2}}$, the length of this rectangle. - The distance between lines (1*) and (2*) is $\frac{2a}{\sqrt{a^2 + 1}}$, the width of this rectangle. The area we seek is $\frac{4a}{\sqrt{1 + a^2}} \cdot \frac{2a}{\sqrt{a^2 + 1}} = \text{(D)} \frac{8a^2}{a^2 + 1}$.

Solution 1.2 (Distance Between Points) The solutions to systems of equations (1) $\cap$ (1*), (1) $\cap$ (2*), (2) $\cap$ (2*), (2) $\cap$ (1*) are

$$(x, y) = \left(\frac{a(a+2)}{a^2+1}, \frac{a(2a-1)}{a^2+1} \right), \left(-\frac{a(a-2)}{a^2+1}, \frac{a(2a+1)}{a^2+1} \right), \left(-\frac{a(a+2)}{a^2+1}, -\frac{a(2a-1)}{a^2+1} \right), \\ \left(\frac{a(a-2)}{a^2+1}, -\frac{a(2a+1)}{a^2+1} \right)$$

respectively, which are the consecutive vertices of this rectangle. By the Distance Formula, the length and width of this rectangle are $\frac{4a\sqrt{a^2+1}}{a^2+1}$ and $\frac{2a\sqrt{a^2+1}}{a^2+1}$, respectively. The area we seek is $\frac{4a\sqrt{a^2+1}}{a^2+1} \cdot \frac{2a\sqrt{a^2+1}}{a^2+1} = \text{(D)} \frac{8a^2}{a^2+1}$

Solution 2:

Similar to Solution 1, we will use the equations from the four cases: 1. $x + ay = 2a$. This is a line with x -intercept $2a$, y -intercept $\frac{2}{a}$, and slope $-\frac{1}{a}$. 2. $x + ay = -2a$. This is a line with x -intercept $-2a$, y -intercept $-\frac{2}{a}$, and slope $-\frac{1}{a}$. 3. $ax - y = a$. This is a line with x -intercept 1 , y -intercept $-a$, and slope a . 4. $ax - y = -a$. This is a line with x -intercept -1 , y -intercept a , and slope a . Let $\tan A = a$. The area of the rectangle created by the four equations can be written as

$$\begin{aligned} 2a \cdot \cos A \cdot 4 \sin A &= 8a \cos A \cdot \sin A \\ &= 8a \cdot \frac{1}{\sqrt{a^2+1}} \cdot \frac{a}{\sqrt{a^2+1}} \\ &= \text{(D)} \frac{8a^2}{a^2+1}. \end{aligned}$$

25 How many ways are there to place **3** indistinguishable red chips, **3** indistinguishable blue chips, and **3** indistinguishable green chips in the squares of a 3×3 grid so that no two chips of the same color are directly adjacent to each other, either vertically or horizontally? (2021 AMC Spring 10A, Question #25)

- A. 12 B. 18 C. 24 D. 30 E. 36

Answer E

Solution Solution 1:

Call the different colors A, B, C . There are $3! = 6$ ways to rearrange these colors to these three letters, so 6 must be multiplied after the letters are permuted in the grid.

WLOG assume that A is in the center. $\begin{matrix} ? & ? & ? \\ ? & A & ? \\ ? & ? & ? \end{matrix}$ In this configuration, there are two

cases, either all the A 's lie on the same diagonal: $\begin{matrix} & ? & ? & A \\ & ? & A & ? \\ A & ? & ? & \end{matrix}$ or all the other two

A 's are on adjacent corners: $\begin{matrix} A & ? & \sim A \\ ? & \sim A & ? \\ ? & ? & ? \end{matrix}$ In the first case there are two ways to order

them since there are two diagonals, and in the second case there are four ways to order them since there are four pairs of adjacent corners. In each case there is only

one way to put the three B 's and the three C 's as shown in the diagrams. $\begin{matrix} C & B & A \\ B & A & C \\ A & C & B \\ A & B & A \\ C & A & C \\ B & C & B \end{matrix}$

This means that there are $4 + 2 = 6$ ways to arrange A, B , and C in the grid, and there are 6 ways to rearrange the colors. Therefore, there are $6 \cdot 6 = 36$ ways in total, which is (E) 36

Solution 2:

Case (1): We have a permutation of R, B , and G as all of the rows. There are $3!$ ways to rearrange these three colors. After finishing the first row, we move onto the second. Notice how the second row must be a derangement of the first one. By the derangement formula, $\frac{3!}{e} \approx 2$, so there are two possible permutations of the second row. (Note: You could have also found the number of derangements of PIE). Finally, there are 2 possible permutations for the last row. Thus, there are $3! \cdot 2 \cdot 2 = 24$ possibilities.

Case (2): All of the rows have two chips that are the same color and one that is different. There are obviously 3 possible configurations for the first row, 2 for the second, and 2 for the third. Thus, there are $3 \cdot 2 \cdot 2 = 12$ possibilities. Therefore, our answer is $24 + 12 = (E) 36$.