

2021 AMC 10A Fall

1 What is the value of $\frac{(2112 - 2021)^2}{169}$? (2021 AMC Fall 10A, Question #1)

- A. 7 B. 21 C. 49 D. 64 E. 91

Answer C

Solution Solution 1 (Laws of Exponents):

$$\text{We have } \frac{(2112 - 2021)^2}{169} = \frac{91^2}{169} = \frac{91^2}{13^2} = \left(\frac{91}{13}\right)^2 = 7^2 = \text{(C)}49$$

Solution 2 (Difference of Squares) :

$$\text{We have } \frac{(2112 - 2021)^2}{169} = \frac{91^2}{169} = \frac{(10^2 - 3^2)^2}{169} = \frac{(13 \cdot 7)^2}{169} = \frac{13^2 \cdot 7^2}{13^2} = 7^2 = \text{(C)}49$$

2 Menkara has a 4×6 index card. If she shortens the length of one side of this card by 1 inch, the card would have area 18 square inches. What would the area of the card be in square inches if instead she shortens the length of the other side by 1 inch? (2021 AMC Fall 10A, Question #2)

- A. 16 B. 17 C. 18 D. 19 E. 20

Answer E

Solution

We construct the following table:

Scenario	Length	Width	Area
Initial	4	6	24
Menkara shortens one side.	3	6	18
Menkara shortens other side instead.	4	5	20

Therefore, the answer is (E) 20.

- 3 What is the maximum number of balls of clay of radius 2 that can completely fit inside a cube of side length 6 assuming the balls can be reshaped but not compressed before they are packed in the cube?(2021 AMC Fall 10A, Question #3)

A. 3 B. 4 C. 5 D. 6 E. 7

Answer D

Solution Solution 1:

The volume of the cube is $V_{\text{cube}} = 6^3 = 216$, and the volume of a clay ball is $V_{\text{ball}} = \frac{4}{3} \cdot \pi \cdot 2^3 = \frac{32}{3}\pi$. Since the balls can be reshaped but not compressed, the maximum number of balls that can completely fit inside a cube is $\left\lfloor \frac{V_{\text{cube}}}{V_{\text{ball}}} \right\rfloor = \left\lfloor \frac{81}{4\pi} \right\rfloor$. Approximating with $\pi \approx 3.14$, we have $12 < 4\pi < 13$, or $\left\lfloor \frac{81}{13} \right\rfloor \leq \left\lfloor \frac{81}{4\pi} \right\rfloor \leq \left\lfloor \frac{81}{12} \right\rfloor$. We simplify to get $6 \leq \left\lfloor \frac{81}{4\pi} \right\rfloor \leq 6$ from which $\left\lfloor \frac{81}{4\pi} \right\rfloor = (\text{D}) 6$.

Solution 2:

As shown in Solution 1, we conclude that the maximum number of balls that can completely fit inside a cube is $\left\lfloor \frac{81}{4\pi} \right\rfloor$. By an underestimation $\pi \approx 3$, we have $4\pi > 12$, or $\frac{81}{4\pi} < 6\frac{3}{4}$. By an overestimation $\pi \approx \frac{22}{7}$, we have $4\pi < \frac{88}{7}$, or $\frac{81}{4\pi} > 6\frac{39}{88}$. Together, we get $6 < 6\frac{39}{88} < \frac{81}{4\pi} < 6\frac{3}{4} < 7$ from which $\left\lfloor \frac{81}{4\pi} \right\rfloor = (\text{D}) 6$.

- 4 Mr. Lopez has a choice of two routes to get to work. Route A is 6 miles long, and his average speed along this route is 30 miles per hour. Route B is 5 miles long, and his average speed along this route is 40 miles per hour, except for a $\frac{1}{2}$ -mile stretch in a school zone where his average speed is 20 miles per hour. By how many minutes is Route B quicker than Route A?(2021 AMC Fall 10A, Question #4)

A. $2\frac{3}{4}$ B. $3\frac{3}{4}$ C. $4\frac{1}{2}$ D. $5\frac{1}{2}$
E. $6\frac{3}{4}$

Answer B

Solution Solution 1:

If Mr. Lopez chooses Route **A**, then he will spend $\frac{6}{30} = \frac{1}{5}$ hour, or $\frac{1}{5} \cdot 60 = 12$ minutes. If Mr. Lopez chooses Route **B**, then he will spend $\frac{9/2}{40} + \frac{1/2}{20} = \frac{11}{80}$ hour, or $\frac{11}{80} \cdot 60 = 8\frac{1}{4}$ minutes. Therefore, Route B is quicker than Route A by $12 - 8\frac{1}{4} = (B) 3\frac{3}{4}$ minutes.

Solution 2:

We use the equation $d = st$ to solve this problem. Recall that 1 mile per hour is equal to $\frac{1}{60}$ mile per minute. For Route A, the distance is 6 miles and the speed to travel this distance is $\frac{1}{2}$ mile per minute. Thus, the time it takes on Route **A** is 12 minutes. For Route **B**, we have to use the equation twice: once for the distance of $5 - \frac{1}{2} = \frac{9}{2}$ miles with a speed of $\frac{2}{3}$ mile per minute and a distance of $\frac{1}{2}$ miles at a speed of $\frac{1}{3}$ mile per minute. Thus, the time it takes to go on Route **B** is $\frac{9}{2} \cdot \frac{3}{2} + \frac{1}{2} \cdot 3 = \frac{27}{4} + \frac{3}{2} = \frac{33}{4}$ minutes. Thus, Route **B** is $12 - \frac{33}{4} = \frac{15}{4} = 3\frac{3}{4}$ faster than Route A. Thus, the answer is (B) $3\frac{3}{4}$.

5 The six-digit number 2020A is prime for only one digit **A**. What is **A** ? (2021 AMC Fall 10A, Question #5)

- A. 1 B. 3 C. 5 D. 7 E. 9

Answer E

Solution Solution 1:

First, modulo 2 or 5, 20210A $\equiv A$. Hence, $A \neq 0, 2, 4, 5, 6, 8$. Second modulo 3, 20210A $\equiv 2 + 0 + 2 + 1 + 0 + A \equiv 5 + A$. Hence, $A \neq 1, 4, 7$. Third, modulo 11, 20210A $\equiv A + 1 + 0 - 0 - 2 - 2 \equiv A - 3$. Hence, $A \neq 3$. Therefore, the answer is **(E) 9**.

Solution 2:

$202100 \implies$ divisible by 2. $202101 \implies$ divisible by 3. $202102 \implies$ divisible by 2.
 $202103 \implies$ divisible by 11. $202104 \implies$ divisible by 2. $202105 \implies$ divisible by 5.
 $202106 \implies$ divisible by 2. $202107 \implies$ divisible by 3. $202108 \implies$ divisible by 2. This
 leaves only $A = (E) 9$.

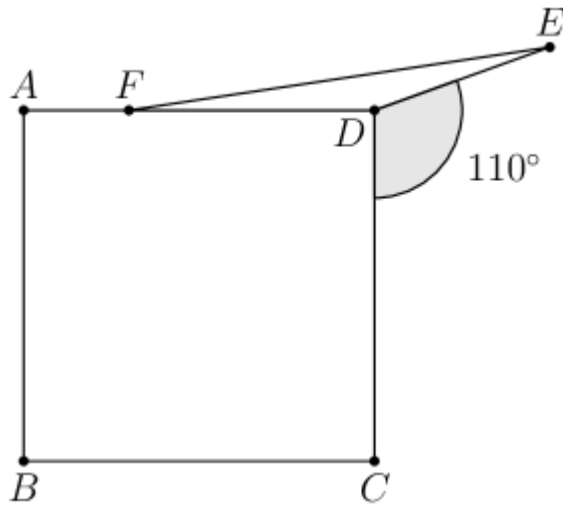
- 6 Elmer the emu takes 44 equal strides to walk between consecutive telephone poles on a rural road. Oscar the ostrich can cover the same distance in 12 equal leaps. The telephone poles are evenly spaced, and the 41st pole along this road is exactly one mile (5280 feet) from the first pole. How much longer, in feet, is Oscar's leap than Elmer's stride?(2021 AMC Fall 10A, Question #6)

A. 6 B. 8 C. 10 D. 11 E. 15

Answer E

Solution There are $41 - 1 = 40$ gaps between the 41 telephone poles, so the distance of each gap is $5280 \div 40 = 132$ feet. Each of Oscar's leaps covers $132 \div 12 = 11$ feet, and each of Elmer's strides covers $132 \div 44 = 3$ feet Therefore, Oscar's leap is $11 - 3 = (\mathbf{B})8$ feet longer than Elmer's stride.

- 7 As shown in the figure below, point E lies on the opposite half-plane determined by line CD from point A so that $\angle CDE = 110^\circ$. Point F lies on $\overline{AD}$ so that $DE = DF$, and $ABCD$ is a square. What is the degree measure of $\angle AFE$?(2021 AMC Fall 10A, Question #7)



- A. 160 B. 164 C. 166 D. 170 E. 174

Answer D

Solution By angle subtraction, we have $\angle ADE = 360^\circ - \angle ADC - \angle CDE = 160^\circ$. Note that $\triangle DEF$ is isosceles, so $\angle DFE = \frac{180^\circ - \angle ADE}{2} = 10^\circ$. Finally, we get $\angle AFE = 180^\circ - \angle DFE =$ (D) 170 degrees.

- 8 A two-digit positive integer is said to be cuddly if it is equal to the sum of its nonzero tens digit and the square of its units digit. How many two-digit positive integers are cuddly?(2021 AMC Fall 10A, Question #8)

- A. 0 B. 1 C. 2 D. 3 E. 4

Answer B

Solution Solution 1:

Note that the number $\underline{xy} = 10x + y$. By the problem statement,
 $10x + y = x + y^2 \implies 9x = y^2 - y \implies 9x = y(y - 1)$. From this we see that $y(y - 1)$ must be divisible by 9. This only happens when $y = 9$. Then, $x = 8$. Thus, there is only **(B)**1 cuddly number, which is 89.

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Solution 2:

If the tens digit is a and the ones digit is b then the number is $10a + b$ so we have the equation $10a + b = a + b^2$. We can guess and check after narrowing the possible cuddly numbers down to 13, 14, 24, 25, 35, 36, 46, 47, 57, 68, 78, 89, and 99. (We can narrow it down to these by just thinking about how a 's value affects b 's value and then check all the possibilities.) Checking all of these we get that there is only (B) 1 2-digit cuddly number, and it is 89.

- 9 When a certain unfair die is rolled, an even number is 3 times as likely to appear as an odd number. The die is rolled twice. What is the probability that the sum of the numbers rolled is even?(2021 AMC Fall 10A, Question #9)

A. $\frac{3}{8}$ B. $\frac{4}{9}$ C. $\frac{5}{9}$ D. $\frac{9}{16}$ E. $\frac{5}{8}$

Answer E

Solution Since an even number is 3 times more likely to appear than an odd number, the probability of an even number appearing is $\frac{3}{4}$. Since the problem states that the sum of the two die must be even, the numbers must both be even or both be odd. We either have EE or OO, so we have $\frac{3}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} + \frac{9}{16} = \frac{10}{16} = (\text{E}) \frac{5}{8}$

- 10 A school has 100 students and 5 teachers. In the first period, each student is taking one class, and each teacher is teaching one class. The enrollments in the classes are 50, 20, 20, 5, and 5. Let t be the average value obtained if a teacher is picked at random and the number of students in their class is noted. Let s be the average value obtained if a student was picked at random and the number of students in their class, including the student, is noted. What is $t - s$?(2021 AMC Fall 10A, Question #10)

A. -18.5 B. -13.5 C. 0 D. 13.5 E. 18.5

Answer B

Solution The formula for expected values is $\text{Expected Value} = \sum (\text{Outcome} \cdot \text{Probability})$. We

$$\begin{aligned}
 t &= 50 \cdot \frac{1}{5} + 20 \cdot \frac{1}{5} + 20 \cdot \frac{1}{5} + 5 \cdot \frac{1}{5} + 5 \cdot \frac{1}{5} \\
 &= (50 + 20 + 20 + 5 + 5) \cdot \frac{1}{5} \\
 \text{have } &= 100 \cdot \frac{1}{5} \\
 &= 20, \qquad \qquad \qquad \text{Therefore, the answer is} \\
 s &= 50 \cdot \frac{50}{100} + 20 \cdot \frac{20}{100} + 20 \cdot \frac{20}{100} + 5 \cdot \frac{5}{100} + 5 \cdot \frac{5}{100} \\
 &= 25 + 4 + 4 + 0.25 + 0.25 \\
 &= 33.5 \\
 t - s &= (\mathbf{B}) - 13.5.
 \end{aligned}$$

- 11** Emily sees a ship traveling at a constant speed along a straight section of a river. She walks parallel to the riverbank at a uniform rate faster than the ship. She counts **210** equal steps walking from the back of the ship to the front. Walking in the opposite direction, she counts **42** steps of the same size from the front of the ship to the back. In terms of Emily's equal steps, what is the length of the ship?(2021 AMC Fall 10A, Question #11)

A. 70 B. 84 C. 98 D. 105 E. 126

Answer A

Solution Solution 1:

Let x be the length of the ship. Then, in the time that Emily walks 210 steps, the ship moves $210 - x$ steps. Also, in the time that Emily walks 42 steps, the ship moves $x - 42$ steps. Since the ship and Emily both travel at some constant rate,

$$\frac{210}{210 - x} = \frac{42}{x - 42}.$$

Dividing both sides by 42 and cross multiplying, we get

$$5(x - 42) = 210 - x, \text{ so } 6x = 420, \text{ and } x = (\mathbf{A}) 70.$$

Solution 2:

Let the speed at which Emily walks be 42 steps per hour. Let the speed at which the ship is moving be s . Walking in the direction of the ship, it takes her 210 steps, or

$$\frac{210}{42} = 5 \text{ hours, to travel. We can create an equation: } d = 5(42 - s), \text{ where } d \text{ is the}$$

length of the ship. Walking in the opposite direction of the ship, it takes her 42 steps, or $42/42 = 1$ hour. We can create a similar equation: $d = 1(42 + s)$. Now we have two variables and two equations. We can equate the expressions for d and solve for s : $210 - 5s = 42 + s$ Therefore, we have $d = 42 + s = (A) 70$.

$$s = 28$$

12 The base-nine representation of the number N is 27,006,000,052 nine. What is the remainder when N is divided by 5?(2021 AMC Fall 10A, Question #12)

- A. 0 B. 1 C. 2 D. 3 E. 4

Answer D

Solution Solution 1:

Recall that $9 \equiv -1 \pmod{5}$. We expand N by the definition of bases:

$$\begin{aligned} N &= 27,006,000,052_9 \\ &= 2 \cdot 9^{10} + 7 \cdot 9^9 + 6 \cdot 9^6 + 5 \cdot 9 + 2 \\ &\equiv 2 \cdot (-1)^{10} + 7 \cdot (-1)^9 + 6 \cdot (-1)^6 + 5 \cdot (-1) + 2 \pmod{5} \\ &\equiv 2 - 7 + 6 - 5 + 2 \pmod{5} \\ &\equiv -2 \pmod{5} \\ &\equiv (D)3 \pmod{5}. \end{aligned}$$

Solution 2:

We need to first convert N into a regular base-10 number:

$$N = 27,006,000,052_9 = 2 \cdot 9^{10} + 7 \cdot 9^9 + 6 \cdot 9^6 + 5 \cdot 9 + 2. \text{ Now, consider how the last}$$

$$\begin{aligned} 9^0 &= 1 \\ 9^1 &= 9 \\ \text{digit of 9 changes with changes of the power of 9 : } 9^2 &= 81 \quad \text{Note that if } x \text{ is odd,} \\ 9^3 &= 729 \\ 9^4 &= 6561 \end{aligned}$$

then $9^x \equiv 4 \pmod{5}$. On the other hand, if x is even, then $9^x \equiv 1 \pmod{5}$. Therefore, we

$$\begin{aligned} N &\equiv 2 \cdot (1) + 7 \cdot (4) + 6 \cdot (1) + 5 \cdot (4) + 2 \cdot (1) \pmod{5} \\ &\equiv 2 + 28 + 6 + 20 + 2 \pmod{5} \\ \text{have } &\equiv 58 \pmod{5} \\ &\equiv (D)3 \pmod{5} \end{aligned} \quad \text{Note that for the odd}$$

case, $9^x \equiv -1 \pmod{5}$ may simplify the process further, as given by Solution 1.

- 13 Each of 6 balls is randomly and independently painted either black or white with equal probability. What is the probability that every ball is different in color from more than half of the other 5 balls?(2021 AMC Fall 10A, Question #13)

A. $\frac{1}{64}$ B. $\frac{1}{6}$ C. $\frac{1}{4}$ D. $\frac{5}{16}$ E. $\frac{1}{2}$

Answer D

Solution Solution 1:

Note that for this restriction to be true, there must be 3 balls of each color. There are a total of $2^6 = 64$ ways to color the balls, and there are $\binom{6}{3} = 20$ ways for three balls chosen to be painted white. Thus, the answer is $\frac{20}{64} = \text{(D)} \frac{5}{16}$

Solution 2:

For this restriction to be upheld, there must be three black and three white balls.

One such way for this to occur is the arrangement *BBBWWWW*, which has a $\frac{1}{2^6}$ probability of occurring. However, there are $\frac{6!}{3! \cdot 3!}$ ways to arrange the three black and three white balls, meaning that the answer is, $\frac{1}{64} \cdot \frac{6!}{3! \cdot 3!} = \text{(D)} \frac{5}{16}$

- 14 How many ordered pairs (x, y) of real numbers satisfy the following system of equations?
- $$\begin{aligned} x^2 + 3y &= 9 \\ (|x| + |y| - 4)^2 &= 1 \end{aligned}$$

A. 1 B. 2 C. 3 D. 5 E. 7

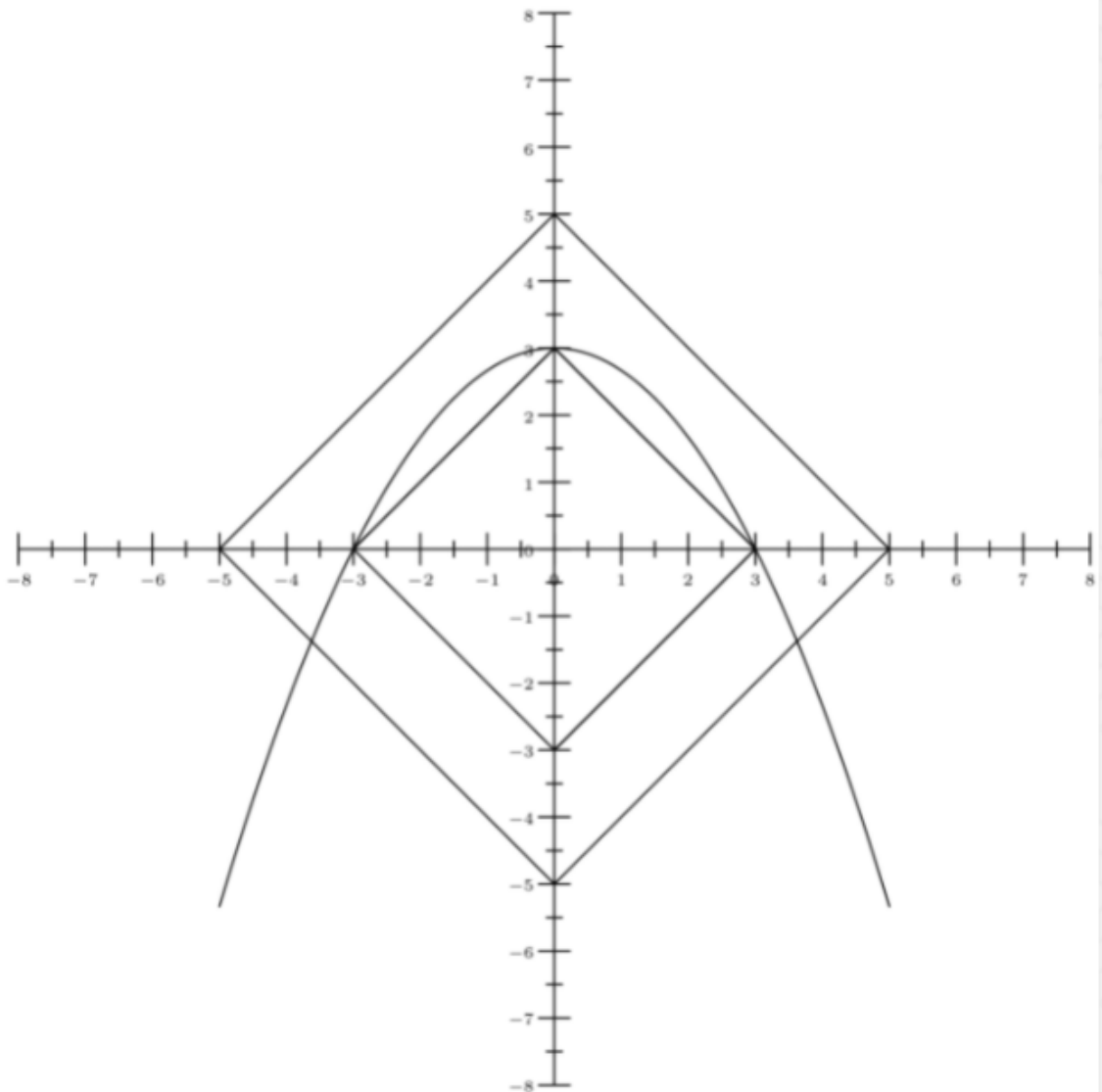
Answer D

Solution Solution 1:

The second equation is $(|x| + |y| - 4)^2 = 1$. We know that the graph of $|x| + |y|$ is a very simple diamond shape, so let's see if we can reduce this equation to that form:

$(|x| + |y| - 4)^2 = 1 \implies |x| + |y| - 4 = \pm 1 \implies |x| + |y| = \{3, 5\}$. We now have two separate graphs for this equation and one graph for the first equation, so let's put it

on the coordinate plane:



Solution 2:

We can manipulate the first equation to get $y = -\frac{x^2}{3} + 3$. From the second equation, we have that $|x| + |y| - 4 = 1$ or $|x| + |y| - 4 = -1$. We will consider each case separately. If $|x| + |y| - 4 = 1$, then $|x| + |y| = 5$. The graph of this is a square with vertices $(5, 0)$, $(-5, 0)$, $(0, 5)$ and $(0, -5)$. The parabola from the first equation is downwards facing, and its vertex is inside this square; the parabola will clearly intersect the square twice. Therefore, this case gives us 2 solutions. If

$|x| + |y| - 4 = -1$, then $|x| + |y| = 3$. The graph of this is a square with vertices $(3, 0)$, $(-3, 0)$, $(0, 3)$ and $(0, -3)$. The vertex of the parabola from the first equation is on one of the corners of this square (in particular, $(0, 3)$). Also, at $y = 0$, the parabola has

x intercepts of ± 3 ; the square passes through both of those points. If we continue to move down, the square narrows in, while the parabola continues to expand.

Therefore, these are our only 3 intersection points in this case: $(0, 3)$, $(3, 0)$ and $(-3, 0)$.

This case gives us 3 solutions. Adding these two cases together, we get our final answer of (D) 5 .

- 15 Isosceles triangle ABC has $AB = AC = 3\sqrt{6}$, and a circle with radius $5\sqrt{2}$ is tangent to line AB at B and to line AC at C . What is the area of the circle that passes through vertices A, B , and C ? (2021 AMC Fall 10A, Question #15)

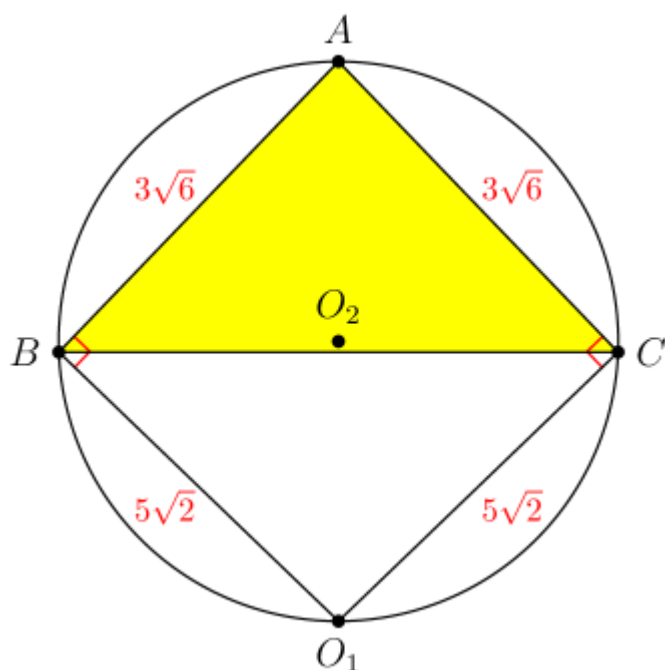
A. 24π B. 25π C. 26π D. 27π E. 28π

Answer C

Solution Solution 1:

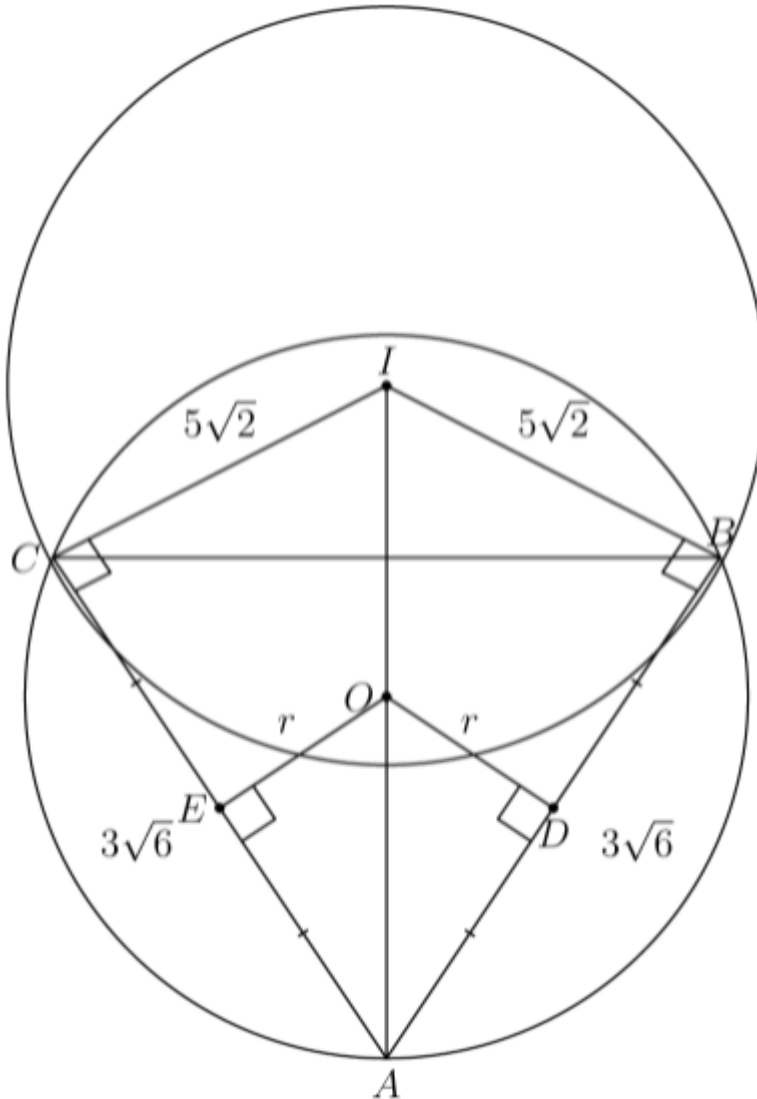
Let $\odot O_1$ be the circle with radius $5\sqrt{2}$ that is tangent to $\overleftrightarrow{AB}$ at B and to $\overleftrightarrow{AC}$ at C .

Note that $\angle ABO_1 = \angle ACO_1 = 90^\circ$. Since the opposite angles of quadrilateral ABO_1C are supplementary, quadrilateral ABO_1C is cyclic. Let $\odot O_2$ be the circumcircle of quadrilateral ABO_1C . It follows that $\odot O_2$ is also the circumcircle of $\triangle ABC$, as shown below:



By the Inscribed Angle Theorem, we conclude that $\overline{AO_1}$ is the diameter of $\odot O_2$. By the Pythagorean Theorem on right $\triangle ABO_1$, we have $AO_1 = \sqrt{AB^2 + BO_1^2} = 2\sqrt{26}$. Therefore, the area of $\odot O_2$ is $\pi \cdot \left(\frac{AO_1}{2}\right)^2 = (\text{C})26\pi$.

Solution 2:



Because circle I is tangent to $\overline{AB}$ at B , $\angle ABI \cong 90^\circ$. Because O is the circumcenter of $\triangle ABC$, $\overline{OD}$ is the perpendicular bisector of $\overline{AB}$, and $\angle BAI \cong \angle DAO$, so therefore $\triangle ADO \sim \triangle ABI$ by AA similarity. Then we have $\frac{AD}{AB} = \frac{DO}{BI} \Rightarrow \frac{1}{2} = \frac{r}{5\sqrt{2}} \Rightarrow r = \frac{5\sqrt{2}}{2}$. We also know that $\overline{AD} = \frac{3\sqrt{6}}{2}$ because of the perpendicular bisector, so the hypotenuse of $\triangle ADO$ is

$$\sqrt{\left(\frac{5\sqrt{2}}{2}\right)^2 + \left(\frac{3\sqrt{6}}{2}\right)^2} = \sqrt{\frac{25}{2} + \frac{27}{2}} = \sqrt{26} \text{ This is the radius of the circumcircle of}$$

$\triangle ABC$, so the area of this circle is **(C)** 26π .

16 The graph of $f(x) = ||\lfloor x \rfloor| - |\lfloor 1 - x \rfloor||$ is symmetric about which of the following? (Here $\lfloor x \rfloor$ is the greatest integer not exceeding x) (2021 AMC Fall 10A, Question #16)

- A. the y -axis
- B. the line $x = 1$
- C. the origin
- D. the point $\left(\frac{1}{2}, 0\right)$
- E. the point $(1, 0)$

Answer D

Solution Solution1:

$$1. \lfloor x + n \rfloor = \lfloor x \rfloor + n \text{ and } \lceil x + n \rceil = \lceil x \rceil + n \quad 2. \lfloor -x \rfloor = -\lceil x \rceil \quad 3. \lceil x \rceil - \lfloor x \rfloor = \begin{cases} 0 & \text{if } x \in \mathbb{Z} \\ 1 & \text{if } x \notin \mathbb{Z} \end{cases}$$

$$f(x) = ||\lfloor x \rfloor| - |\lfloor 1 - x \rfloor||$$

We rewrite $f(x)$ as $= ||\lfloor x \rfloor| - |-\lceil x - 1 \rceil||$ We apply casework to the value of x : 1.

$$= ||\lfloor x \rfloor| - |-\lceil x \rceil + 1||$$

$x \in \mathbb{Z}^-$ It follows that $f(x) = -x - (-x + 1) = -1$. 2. $x = 0$ It follows that

$f(x) = 0 - 1 = -1$. 3. $x \in \mathbb{Z}^+$ It follows that $f(x) = x - (x - 1) = 1$. 4. $x \notin \mathbb{Z}$ and $x < 0$ It

follows that $f(x) = -\lfloor x \rfloor - (-\lceil x \rceil + 1) = (\lceil x \rceil - \lfloor x \rfloor) - 1 = 0$. 5. $x \notin \mathbb{Z}$ and $0 < x < 1$ It

follows that $f(x) = 0 - 0 = 0$. 6. $x \notin \mathbb{Z}$ and $x > 1$ It follows that

$f(x) = \lfloor x \rfloor - (\lceil x \rceil - 1) = (\lfloor x \rfloor - \lceil x \rceil) + 1 = 0$. Together, we have

$$f(x) = \begin{cases} -1 & \text{if } x \in \mathbb{Z}^- \cup \{0\} \\ 1 & \text{if } x \in \mathbb{Z}^+ \\ 0 & \text{if } x \notin \mathbb{Z} \end{cases} \quad \text{so the graph of } f(x) \text{ is symmetric about (D) the point}$$

$\left(\frac{1}{2}, 0\right)$. Alternatively, we can eliminate (A), (B), (C), and (E) once we finish with Case 3

. This leaves us with (D).

Solution 2:

Denote $x = a + b$, where $a \in \mathbb{Z}$ and $b \in [0, 1)$. Hence, a is the integer part of x and b is the decimal part of x .

Case 1: $b = 0$. We have $f(x) = ||\lfloor x \rfloor| - |\lfloor 1 - x \rfloor|| = |a| - |1 - a|$

$$= \begin{cases} a - (a - 1) & \text{if } a \in \mathbb{Z} \text{ and } a \geq 1 \\ -1 & \text{if } a = 0 \\ -a - (1 - a) & \text{if } a \in \mathbb{Z} \text{ and } a \leq -1 \end{cases} = \begin{cases} 1 & \text{if } a \in \mathbb{Z} \text{ and } a \geq 1 \\ -1 & \text{if } a = 0 \\ -1 & \text{if } a \in \mathbb{Z} \text{ and } a \leq -1 \end{cases}$$

$$= \begin{cases} 1 & \text{if } a \in \mathbb{Z} \text{ and } a \geq 1 \\ -1 & \text{if } a \in \mathbb{Z} \text{ and } a \leq 0 \end{cases}$$

$$f(x) = ||x| - |1-x||$$

$$= |a| - ||1-a-b||$$

Case 2: $b \neq 0$. We have $= |a| - ||-a + (1-b)||$ Therefore, the graph of $f(x)$ is

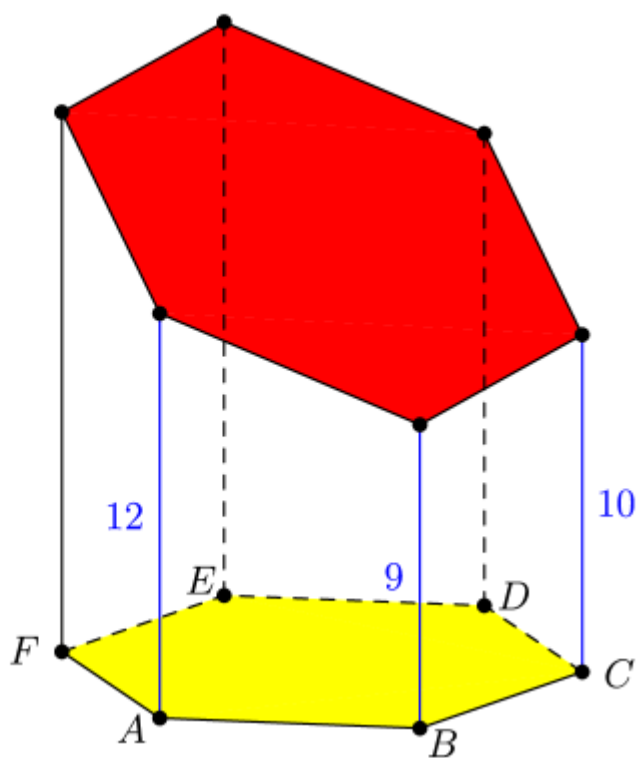
$$= |a| - |-a|$$

$$= 0.$$

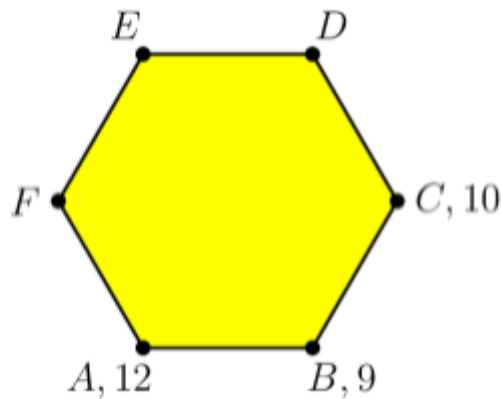
symmetric through the point $\left(\frac{1}{2}, 0\right)$. Therefore, the answer is (D) the point $\left(\frac{1}{2}, 0\right)$

- 17 An architect is building a structure that will place vertical pillars at the vertices of regular hexagon $ABCDEF$, which is lying horizontally on the ground. The six pillars will hold up a flat solar panel that will not be parallel to the ground. The heights of pillars at A, B , and C are 12, 9, and 10 meters, respectively. What is the height, in meters, of the pillar at E ? (2021 AMC Fall 10A, Question #17)

Three-Dimensional Diagram



Two-Dimensional Diagram



- A. 9 B. $6\sqrt{3}$ C. $8\sqrt{3}$ D. 17 E. $12\sqrt{3}$

Answer D

Solution Solution 1:

The pillar at B has height 9 and the pillar at A has height 12. Since the solar panel is flat, the inclination from pillar B to pillar A is 3° . Call the center of the hexagon G . Since $\overrightarrow{CG} \parallel \overrightarrow{BA}$, it follows that the solar panel has height 13 at G . Since the solar panel is flat, the heights of the solar panel at B, G , and E are collinear. Therefore, the pillar at E has height $9 + 4 + 4 =$ (D) 17.

Solution 2:

Let the height of the pillar at D be x . Notice that the difference between the heights of pillar C and pillar D is equal to the difference between the heights of pillar A and pillar F . So, the height at F is $x + 2$. Now, doing the same thing for pillar E we get the height is $x + 3$. Therefore, we can see the difference between the heights at pillar C and pillar D is half the difference between the heights at B and E , so

$$\begin{aligned} x + 3 - 9 &= 2 \cdot (x - 10) \\ x - 6 &= 2 \cdot (x - 10) \text{ The answer is } x + 3 = \text{(D) } 17. \\ x &= 14 \end{aligned}$$

- 18 A farmer's rectangular field is partitioned into 2 by 2 grid of 4 rectangular sections as shown in the figure. In each section the farmer will plant one crop: corn, wheat,

soybeans, or potatoes. The farmer does not want to grow corn and wheat in any two sections that share a border, and the farmer does not want to grow soybeans and potatoes in any two sections that share a border. Given these restrictions, in how many ways can the farmer choose crops to plant in each of the four sections of the field?(2021 AMC Fall 10A, Question #18)

- A. 12 B. 64 C. 84 D. 90 E. 144

Answer C

Solution Solution 1:

There are 4 possibilities for the top-left section. It follows that the top-right and bottom-left sections each have 3 possibilities, so they have $3^2 = 9$ combinations. We have two cases:

1. The top-right and bottom-left sections have the same crop. Note that 3 of the 9 combinations of the top-right and bottom-left sections satisfy this case, from which the bottom-right section has 3 possibilities. Therefore, there are $4 \cdot 3 \cdot 3 = 36$ ways in this case.

2. The top-right and bottom-left sections have different crops. Note that 6 of the 9 combinations of the top-right and bottom-left sections satisfy this case, from which the bottom-right section has 2 possibilities. Therefore, there are $4 \cdot 6 \cdot 2 = 48$ ways in this case. Together, the answer is $36 + 48 = (C) 84$.

Solution 2:

We will do casework on the type of crops in the field.

Case 1: all of a kind. If all four sections have the same type of crop, there are simply 4 ways to choose crops for the sections.

Case 2: 3 of a kind, 1 of another kind. Since the one of another kind must be adjacent to two of the other crops, when choosing the type of crops in this case, we cannot choose soybeans and potatoes, or corn and wheat. Therefore, there are $4 \cdot 3 - 2 \cdot 2 = 8$ choices for the two crops we choose for the section (notice we did not

choose by 2 , since the crop we pick first will be the unique one), and 4 ways to choose which section the unique crop is planted on. This gives us a total of $4 \cdot 8 = \underline{32}$ ways to choose crops for the sections.

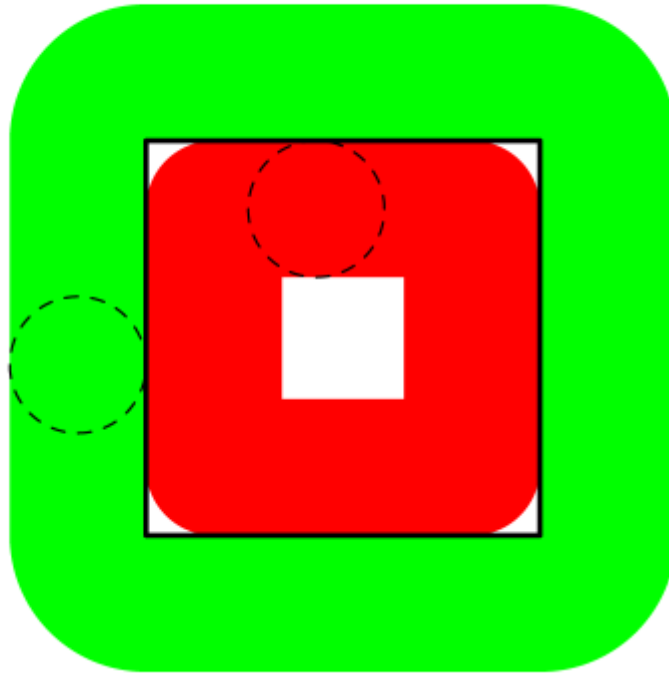
Case 3: 2 of a kind, 2 of another kind. We cannot choose corn and wheat, or soybeans and potatoes, once again, because if we do, the two would have to be adjacent in some way, which the problem disallows. So, there are $\binom{4}{2} - 2 = 4$ ways to choose our two crops (notice that we did choose by 2 , since there are two of both crops). There are $\binom{4}{2} = 6$ ways to choose where one of the crops go, so there are $4 \cdot 6 = \underline{24}$ ways to choose crops for the sections.

Case 4: 2 of a kind, 1 of another kind, 1 of another kind. In cases 2 and 3, we excluded the possibility of choosing bad pairs for our crops (i.e. soybeans and potatoes, or corn and wheat). In this case, it is inevitable that we choose a bad pair, because we are choosing 3 crops this time. The two sections of the same kind must contain the crop that is not part of the bad pair in the trio: for example, if we choose corn, soybeans and potatoes as our three crop types, nor soybeans and potatoes can be the type which occupies two sections in this case; corn must be the one to do so. There are 4 ways to choose the crop that is not part of the bad pair, and then 1 way to choose the bad pair, giving us $4 \cdot 1 = 4$ ways to choose the crops. To separate the bad pair of crops, the two of a kind must be diagonally placed. There are 2 ways to choose where the two of a kind go, and 2 ways to choose which of the bad pair goes where, giving us $2 \cdot 2 = 4$ ways to choose the positions for the crops. In total, there are $4 \cdot 4 = \underline{16}$ ways to choose crops for the sections.

Case 5: every single crop. Bad pairs must be on the same diagonal, so there are 2 ways to choose which pair gets which diagonal, and $2 \cdot 2 = 4$ ways to choose which of each pair goes where on the diagonal, giving us $2 \cdot 4 = \underline{8}$ ways to choose crops for the sections. Adding up all our values, we get our final answer of $4 + 32 + 24 + 16 + 8 = \text{(C) } 84$.

A disk of radius 1 rolls all the way around the inside of a square of side length $s > 4$ and sweeps out a region of area A . A second disk of radius 1 rolls all the way around the outside of the same square and sweeps out a region of area $2A$. The value of s can be written as $a + \frac{b\pi}{c}$, where a, b , and c are positive integers and b and c are relatively prime. What is $a + b + c$? (2021 AMC Fall 10A, Question #19)

Diagram



- A. 10 B. 11 C. 12 D. 13 E. 14

Answer A

Solution The side length of the inner square traced out by the disk with radius 1 is $s - 4$.

However, there is a piece at each corner (bounded by two line segments and one 90° arc) where the disk never sweeps out. The combined area of these four pieces is

$(1 + 1)^2 - \pi \cdot 1^2 = 4 - \pi$. As a result, we have $A = s^2 - (s - 4)^2 - (4 - \pi) = 8s - 20 + \pi$.

Now, we consider the second disk. The part it sweeps is comprised of four quarter circles with radius 2 and four rectangles with side lengths of 2 and s . When we add it all together, we have $2A = 8s + 4\pi$, or $A = 4s + 2\pi$. We equate the expressions for A , and then solve for s : $8s - 20 + \pi = 4s + 2\pi$. We get $s = 5 + \frac{\pi}{4}$, so the answer is

$$5 + 1 + 4 = (A) 10 .$$

20 How many ordered pairs of positive integers (b, c) exist where both $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$ do not have distinct, real solutions?(2021 AMC Fall 10A, Question #20)

- A. 4 B. 6 C. 8 D. 10 E. 12

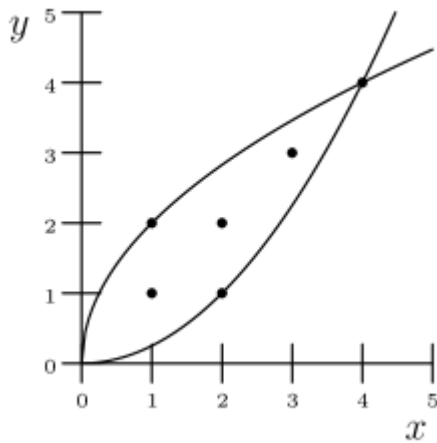
Answer B

Solution Solution 1:

A quadratic equation does not have real solutions if and only if the discriminant is nonpositive. We conclude that: 1. Since $x^2 + bx + c = 0$ does not have real solutions, we have $b^2 \leq 4c$. 2. Since $x^2 + cx + b = 0$ does not have real solutions, we have $c^2 \leq 4b$. Squaring the first inequality, we get $b^4 \leq 16c^2$. Multiplying the second inequality by 16, we get $16c^2 \leq 64b$. Combining these results, we get $b^4 \leq 16c^2 \leq 64b$. We apply casework to the value of b : - If $b = 1$, then $1 \leq 16c^2 \leq 64$, from which $c = 1, 2$. - If $b = 2$, then $16 \leq 16c^2 \leq 128$, from which $c = 1, 2$. - If $b = 3$, then $81 \leq 16c^2 \leq 192$, from which $c = 3$. - If $b = 4$, then $256 \leq 16c^2 \leq 256$, from which $c = 4$. Together, there are **(B)6** ordered pairs (b, c) , namely $(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)$, and $(4, 4)$.

Solution 2:

Similar to Solution 1, use the discriminant to get $b^2 \leq 4c$ and $c^2 \leq 4b$. These can be rearranged to $c \geq \frac{1}{4}b^2$ and $b \geq \frac{1}{4}c^2$. Now, we can roughly graph these two inequalities, letting one of them be the x axis and the other be y . The graph of solutions should be above the parabola and under its inverse, meaning we want points on the graph or in the first area enclosed by the two graphs:



We are looking for lattice points (since b and c are positive integers), of which we can count (B) 6 .

- 21 Each of the 20 balls is tossed independently and at random into one of the 5 bins. Let p be the probability that some bin ends up with 3 balls, another with 5 balls, and the other three with 4 balls each. Let q be the probability that every bin ends up with 4 balls. What is $\frac{p}{q}$?(2021 AMC Fall 10A, Question #21)

A. 1 B. 4 C. 8 D. 12 E. 16

Answer E

Solution Solution 1:

For simplicity purposes, we assume that the balls are indistinguishable and the bins are distinguishable. Let d be the number of ways to distribute 20 balls into 5 bins.

We have $p = \frac{5 \cdot 4 \cdot \binom{20}{3, 5, 4, 4, 4}}{d}$ and $q = \frac{\binom{20}{4, 4, 4, 4, 4}}{d}$ Therefore, the answer is

$$\frac{p}{q} = \frac{5 \cdot 4 \cdot \binom{20}{3, 5, 4, 4, 4}}{\binom{20}{4, 4, 4, 4, 4}} = \frac{5 \cdot 4 \cdot \frac{20!}{3! \cdot 5! \cdot 4! \cdot 4! \cdot 4!}}{\frac{20!}{4! \cdot 4! \cdot 4! \cdot 4! \cdot 4!}} = \frac{5 \cdot 4 \cdot (4! \cdot 4! \cdot 4! \cdot 4! \cdot 4!)}{3! \cdot 5! \cdot 4! \cdot 4! \cdot 4!} = \frac{5 \cdot 4 \cdot 4}{5} = (\text{E})16$$

Remark By the stars and bars argument, we get $d = \binom{20+5-1}{5-1} = \binom{24}{4}$

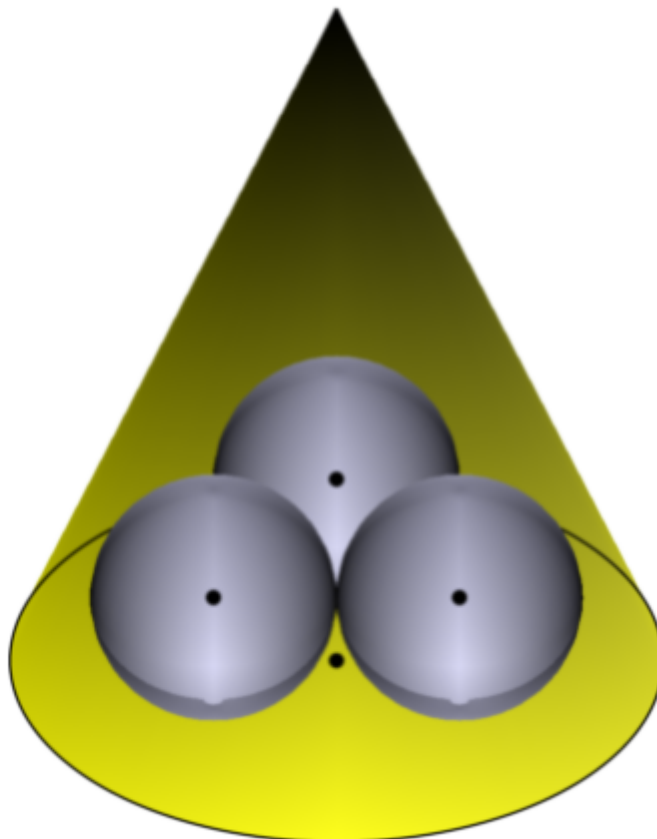
Solution 2:

For simplicity purposes, the balls are indistinguishable and the bins are distinguishable. Let q be equal to $\frac{x}{a}$ where a is the total number of combinations and x is the number of cases where every bin ends up with 4 balls. Notice that we can take 1 ball from one bin and place it in another bin so that some bin ends up with 3 balls, another with 5 balls, and the other three with 4 balls each. We have

$$x \cdot \frac{\binom{5}{1} \cdot \binom{4}{1} \cdot \binom{4}{1}}{5} = 16x \text{ Therefore, we get } p = \frac{16x}{a}, \text{ from which } \frac{p}{q} = \text{(E)}16$$

- 22 Inside a right circular cone with base radius 5 and height 12 are three congruent spheres with radius r . Each sphere is tangent to the other two spheres and also tangent to the base and side of the cone. What is r ? (2021 AMC Fall 10A, Question #22)

Diagram

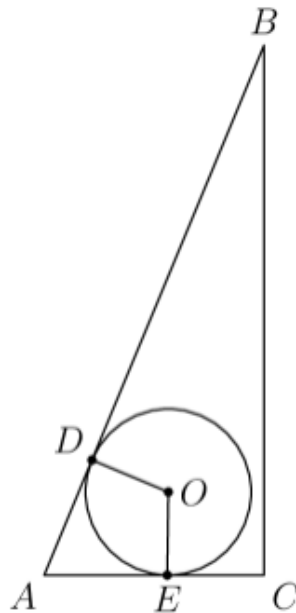


- | | |
|------------------|----------------------------------|
| A. $\frac{3}{2}$ | B. $\frac{90 - 40\sqrt{3}}{11}$ |
| C. 2 | D. $\frac{144 - 25\sqrt{3}}{44}$ |
| E. $\frac{5}{2}$ | |

Answer B

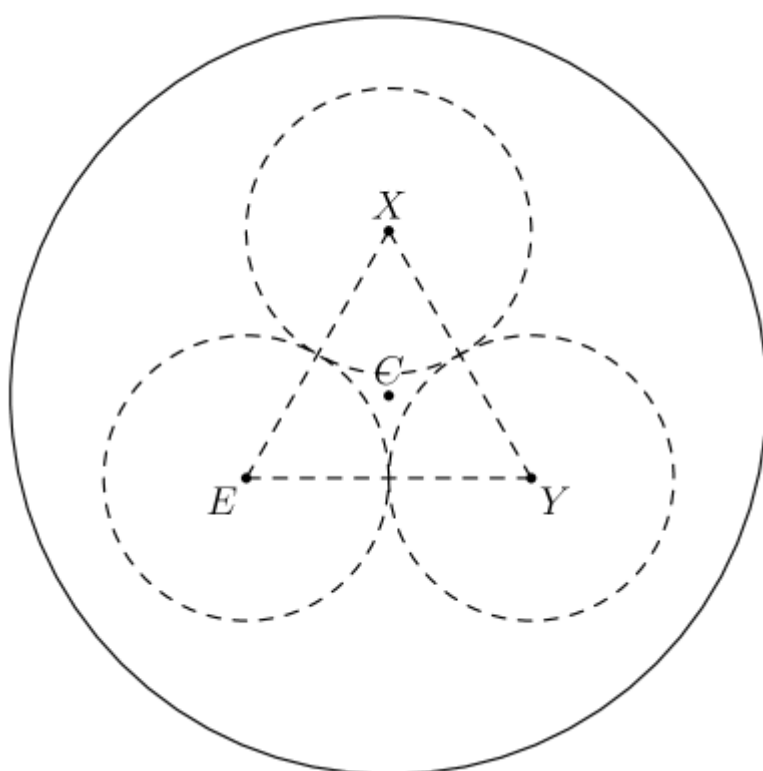
Solution Solution 1:

We can take half of a cross section of the sphere, as such:

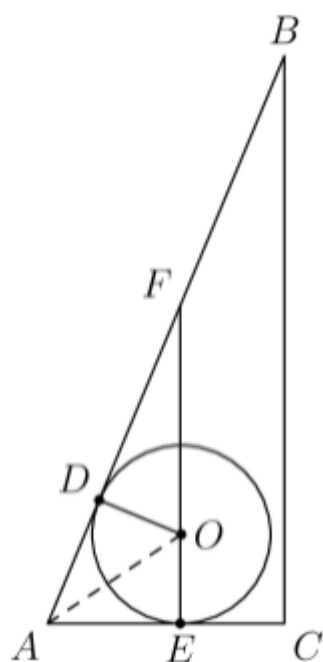


Notice that we chose a cross section where one of the spheres was tangent to the lateral surface of the cone at D .

Notice that we chose a cross section where one of the spheres was tangent to the lateral surface of the cone at D . To evaluate r , we will find AE and EC in terms of r ; we also know that $AE + EC = 5$, so with this, we can solve r . Firstly, to find EC , we can take a bird's eye view of the cone:



Note that C is the centroid of equilateral triangle XEY . Also, since all of the medians of an equilateral triangle are also altitudes, we want to find two-thirds of the altitude from E to XY ; this is because medians cut each other into a 2 to 1 ratio. This equilateral triangle has a side length of $2r$, therefore it has an altitude of length $r\sqrt{3}$; two thirds of this is $\frac{2r\sqrt{3}}{3}$, so $EC = \frac{2r\sqrt{3}}{3}$



To evaluate AE in terms of r , we will extend $\overline{OE}$ past point O to $\overline{AB}$ at point F . $\triangle AEF$ is similar to $\triangle ACB$. Also, AO is the angle bisector of $\angle EAB$. Therefore, by the angle bisector theorem, $\frac{OE}{OF} = \frac{AE}{AF} = \frac{5}{13}$. Also, $OE = r$, so $\frac{r}{OF} = \frac{5}{13}$, so $OF = \frac{13r}{5}$. This means that $AE = \frac{5 \cdot EF}{12} = \frac{5 \cdot (OE + OF)}{12} = \frac{5 \cdot (r + \frac{13r}{5})}{12} = \frac{18r}{12} = \frac{3r}{2}$. We have that $EC = \frac{2r\sqrt{3}}{3}$ and that $AE = \frac{3r}{2}$, so $AC = EC + AE = \frac{2r\sqrt{3}}{3} + \frac{3r}{2} = \frac{4r\sqrt{3} + 9r}{6}$. We also were given that $AC = 5$. Therefore, we have $\frac{4r\sqrt{3} + 9r}{6} = 5$. This is a simple linear equation in terms of r . We can solve for r to get $r = \text{(B)} \frac{90 - 40\sqrt{3}}{11}$.

Solution 2:

We will use coordinates. WLOG, let the coordinates of the center of the base of the cone be the origin. Then, let the center of one of the spheres be $(0, \frac{2r}{\sqrt{3}}, r)$. Note that the distance between this point and the plane given by $12y + 5z = 60$ is r . Thus, by the point-to-plane distance formula, we have $\frac{|12 \cdot \frac{2r}{\sqrt{3}} + 5r - 60|}{\sqrt{0^2 + 5^2 + 12^2}} = r$. Solving for r yields $r = \text{(B)} \frac{90 - 40\sqrt{3}}{11}$.

23 For each positive integer n , let $f_1(n)$ be twice the number of positive integer divisors of n , and for $j \geq 2$, let $f_j(n) = f_1(f_{j-1}(n))$. For how many values of $n \leq 50$ is $f_{50}(n) = 12$? (2021 AMC Fall 10A, Question #23)

- A. 7 B. 8 C. 9 D. 10 E. 11

Answer D

Solution Solution 1:

First, we can test values that would make $f(x) = 12$ true. For this to happen x must have 6 divisors, which means its prime factorization is in the form pq^2 or p^5 , where p and q are prime numbers. Listing out values less than 50 which have these prime factorizations, we find 12, 20, 28, 44, 18, 45, 50 for pq^2 , and just 32 for p^5 . Here 12

especially catches our eyes, as this means if one of $f_i(n) = 12$, each of $f_{i+1}(n), f_{i+2}(n), \dots$ will all be 12. This is because $f_{i+1}(n) = f(f_i(n))$ (as given in the problem statement), so were $f_i(n) = 12$, plugging this in we get $f_{i+1}(n) = f(12) = 12$, and thus the pattern repeats. Hence, as long as for a i , such that $i \leq 50$ and $f_i(n) = 12, f_{50}(n) = 12$ must be true, which also immediately makes all our previously listed numbers, where $f(x) = 12$, possible values of n .

We also know that if $f(x)$ were to be any of these numbers, x would satisfy $f_{50}(n)$ as well. Looking through each of the possibilities aside from 12, we see that $f(x)$ could only possibly be equal to 20 and 18, and still have x less than or equal to 50. This would mean x must have 10, or 9 divisors, and testing out, we see that x will then be of the form p^4q , or p^2q^2 . The only two values less than or equal to 50 would be 48 and 36 respectively. From here there are no more possible values, so tallying our possibilities we count (D) 10 values (Namely 12, 20, 28, 44, 18, 45, 50, 32, 36, 48).

Solution 2:

Observation 1: $f_1(12) = 12$. Hence, if n has the property that $f_j(n) = 12$ for some j , then $f_k(n) = 12$ for all $k > j$.

Observation 2: $f_1(8) = 8$. Hence, if n has the property that $f_j(n) = 8$ for some j , then $f_k(n) = 8$ for all $k > j$.

Case 1: $n = 1$ We have $f_1(n) = 2, f_2(n) = f_1(2) = 4, f_3(n) = f_1(4) = 6, f_4(n) = f_1(6) = 8$. Hence, Observation 2 implies $f_{50}(n) = 8$.

Case 2: n is prime. We have $f_1(n) = 4, f_2(n) = f_1(4) = 6, f_3(n) = f_1(6) = 8$. Hence, Observation 2 implies $f_{50}(n) = 8$.

Case 3 : The prime factorization of n takes the form p_1^2 . We have $f_1(n) = 6, f_2(n) = f_1(6) = 8$. Hence, Observation 2 implies $f_{50}(n) = 8$.

Case 4 : The prime factorization of n takes the form p_1^3 . We have $f_1(n) = 8$. Hence, Observation 2 implies $f_{50}(n) = 8$.

Case 5 : The prime factorization of n takes the form p_1^4 . We have $f_1(n) = 10, f_2(n) = f_1(10) = 8$. Hence, Observation 2 implies $f_{50}(n) = 8$.

Case 6 : The prime factorization of n takes the form p_1^5 . We have $f_1(n) = 12$. Hence, Observation 1 implies $f_{50}(n) = 12$. In this case the only n is $2^5 = 32$.

Case 7: The prime factorization of n takes the form $p_1 p_2$. We have $f_1(n) = 8$. Hence, Observation 2 implies $f_{50}(n) = 8$.

Case 8 : The prime factorization of n takes the form $p_1 p_2^2$. We have $f_1(n) = 12$. Hence, Observation 1 implies $f_{50}(n) = 12$. In this case, all n are 18, 50, 12, 20, 45, 28, 44.

Case 9 : The prime factorization of n takes the form $p_1 p_2^3$. We have $f_1(n) = 16, f_2(n) = f_1(16) = 10, f_3(n) = f_1(10) = 8$. Hence, Observation 2 implies $f_{50}(n) = 8$.

Case 10: The prime factorization of n takes the form $p_1 p_2^4$. We have $f_1(n) = 20, f_2(n) = f_1(20) = 12$. Hence, Observation 1 implies $f_{50}(n) = 12$. In this case, the only n is 48 .

Case 11: The prime factorization of n takes the form $p_1^2 p_2^2$. We have $f_1(n) = 18, f_2(n) = f_1(18) = 12$. Hence, Observation 1 implies $f_{50}(n) = 12$. In this case, the only n is 36 .

Case 12: The prime factorization of n takes the form $p_1 p_2 p_3$. We have $f_1(n) = 16, f_2(n) = f_1(16) = 10, f_3(n) = f_2(10) = 8$. Hence, Observation 2 implies $f_{50}(n) = 8$.

Putting all cases together, the number of feasible $n \leq 50$ is (D) 10

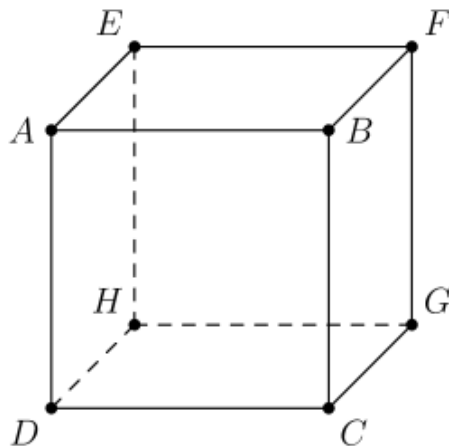
24 Each of the 12 edges of a cube is labeled 0 or 1 . Two labelings are considered different even if one can be obtained from the other by a sequence of one or more rotations and/or reflections. For how many such labelings is the sum of the labels on the edges of each of the 6 faces of the cube equal to 2 ?(2021 AMC Fall 10A, Question #24)

- A. 8 B. 10 C. 12 D. 16 E. 20

Answer E

Solution Solution 1:

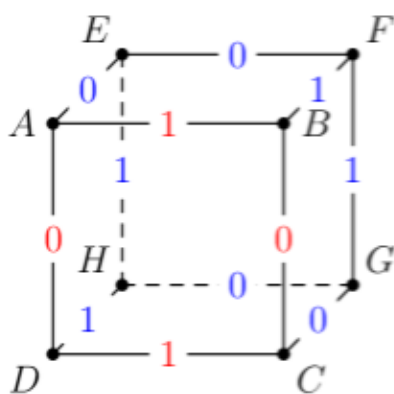
For simplicity purposes, we name this cube $ABCDEFGH$ by vertices, as shown below.



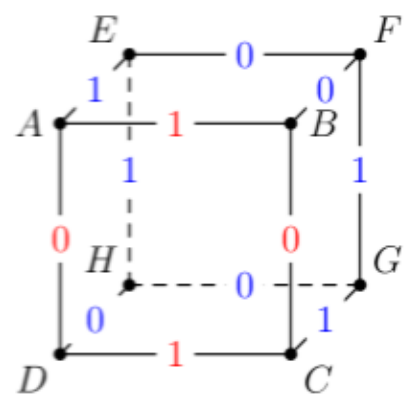
Note that for each face of this cube, two edges are labeled 0 and two edges are labeled 1. For all twelve edges of this cube, we conclude that six edges are labeled 0, and six edges are labeled 1.

We apply casework to face $ABCD$. Recall that there are $\binom{4}{2} = 6$ ways to label its edges:

1. Opposite edges have the same label. There are 2 ways to label the edges of $ABCD$. We will consider one of the ways, then multiply the count by 2. Without the loss of generality, we assume that $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ are labeled 1, 0, 1, 0, respectively: We apply casework to the label of $\overline{AE}$, as shown below.



The label of $\overline{AE}$ is 0.



The label of $\overline{AE}$ is 1.

We have $2 \cdot 2 = 4$ such labelings for this case.

2. Opposite edges have different labels. There are 4 ways to label the edges of $ABCD$. We will consider one of the ways, then multiply the count by 4. Without the loss of generality, we assume that $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ are labeled 1, 1, 0, 0, respectively:

We apply casework to the labels of $\overline{AE}$ and $\overline{BF}$, as shown below.

We have $4 \cdot 4 = 16$ such labelings for this case. Therefore, we have $4 + 16 = \text{(E)} 20$ such labelings in total.

Solution 2:

Since we want the sum of the edges of each face to be 2, we need there to be two 1s and two 0s on each face. Through experimentation, we find that either 2, 4, or all of them have 1-s adjacent to 1s and 0s adjacent to 0 on each face. WLOG, let the first face (counterclockwise) be 0, 0, 1, 1. In this case we are trying to have all of them be adjacent to each other. First face: 0, 0, 1, 1. Second face: 2 choices: 1, 0, 0, 1 or 0, 0, 1, 1. After that, it is basically forced and everything will fall in to place. Since we assumed WLOG, we need to multiply 2 by 4 to get a total of 8 different arrangements.

Secondly, 4 of the faces have all of them adjacent and 2 of the faces do not: WLOG counting counterclockwise, we have 0, 0, 1, 1. Then, we choose the other face next to it. There are two cases, which are 0, 1, 0, 1 and 1, 0, 1, 0. Therefore, this subcase has 4 different arrangements. Then, we can choose the face at front to be 1, 0, 1, 0. This has 4 cases. The sides can either be 0, 1, 1, 0 or 1, 1, 0, 0. Therefore, we have another 8 cases. Summing these up, we have $8 + 4 + 8 = 20$. Therefore, our answer is **(E)20**.

Remark It is very easy to get disorganized when counting, so when doing this problem, make sure to draw a diagram of the cube. Labeling is a bit harder, since we often confuse one side with another. Try doing the problem by labeling sides on the lines (literally letting the lines pass through your 0-s and 1-s.) I found that to be very helpful when solving this problem.

- 25 A quadratic polynomial with real coefficients and leading coefficient 1 is called disrespectful if the equation $p(p(x)) = 0$ is satisfied by exactly three real numbers. Among all the disrespectful quadratic polynomials, there is a unique such polynomial $\tilde{p}(x)$ for which the sum of the roots is maximized. What is $\tilde{p}(1)$? (2021 AMC Fall 10A, Question #25)

A. $\frac{5}{16}$

B. $\frac{1}{2}$

C. $\frac{5}{8}$

D. 1

E. $\frac{9}{8}$

Answer A

Solution Solution 1:

Let r_1 and r_2 be the roots of $\tilde{p}(x)$. Then, $\tilde{p}(x) = (x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1r_2$

. The solutions to $\tilde{p}(\tilde{p}(x)) = 0$ is the union of the solutions to

$$\tilde{p}(x) - r_1 = x^2 - (r_1 + r_2)x + (r_1r_2 - r_1) = 0 \text{ and}$$

$$\tilde{p}(x) - r_2 = x^2 - (r_1 + r_2)x + (r_1r_2 - r_2) = 0. \text{ Note that one of these two quadratics}$$

has one solution (a double root) and the other has two as there are exactly three solutions. WLOG, assume that the quadratic with one root is

$$x^2 - (r_1 + r_2)x + (r_1r_2 - r_1) = 0. \text{ Then, the discriminant is } 0, \text{ so } (r_1 + r_2)^2 = 4r_1r_2 - 4r_1$$

. Thus, $r_1 - r_2 = \pm 2\sqrt{-r_1}$, but for $x^2 - (r_1 + r_2)x + (r_1r_2 - r_2) = 0$ to have two

solutions, it must be the case that $r_1 - r_2 = -2\sqrt{-r_1}$. It follows that the sum of the

roots of $\tilde{p}(x)$ is $2r_1 + 2\sqrt{-r_1}$, whose maximum value occurs when $r_1 = -\frac{1}{4}$. Solving for

r_2 yields $r_2 = \frac{3}{4}$. Therefore, $\tilde{p}(x) = x^2 - \frac{1}{2}x - \frac{3}{16}$, so $\tilde{p}(1) = \text{(A)} \frac{5}{16}$. Remark * For

$x^2 - (r_1 + r_2)x + (r_1r_2 - r_2) = 0$ to have two solutions, the discriminant

$$(r_1 + r_2)^2 - 4r_1r_2 + 4r_2 \text{ must be positive. From here, we get that } (r_1 - r_2)^2 > -4r_2, \text{ so}$$

$$-4r_1 > -4r_2 \implies r_1 < r_2. \text{ Hence, } r_1 - r_2 \text{ is negative, so } r_1 - r_2 = -2\sqrt{-r_1}.$$

Solution 2:

Let $p(x) = (x - h)^2 + k$ for some real constants h and k . Suppose that $p(x)$ has real

roots r and s . Since $p(p(x)) = 0$, we conclude that $p(x) = r$ or $p(x) = s$. Without the

loss of generality, we assume that $p(x) = r$ has two real solutions and $p(x) = s$ has

one real solution. Therefore, we have $k = s$, from which $p(x) = (x - h)^2 + s$. As $p(s) = 0$

$$, \text{ we expand the left side to obtain } (s - h)^2 + s = 0, \text{ or } s^2 - (2h - 1)s + h^2 = 0. \quad (*)$$

Since $(*)$ has real solutions for s , the discriminant is nonnegative: $(2h - 1)^2 - 4h^2 \geq 0$.

We solve this inequality to get $h \leq \frac{1}{4}$. Either by the axis of symmetry or Vieta's

Formulas, note that $r + s = 2h$. As we wish to maximize $2h$, we maximize h .

Substituting $h = \frac{1}{4}$ into $(*)$, we obtain $s^2 + \frac{1}{2}s + \frac{1}{16} = 0$. We factor the left side to get

$\left(s + \frac{1}{4}\right)^2 = 0$, or $s = -\frac{1}{4}$. Finally, the unique such polynomial is $\tilde{p}(x) = \left(x - \frac{1}{4}\right)^2 - \frac{1}{4}$
from which $\tilde{p}(1) = \text{(A)} \frac{5}{16}$