

2020 AMC 8

- 1 Luka is making lemonade to sell at a school fundraiser. His recipe requires 4 times as much water as sugar and twice as much sugar as lemon juice. He uses 3 cups of lemon juice. How many cups of water does he need?

A. 6 B. 8 C. 12 D. 18 E. 24

Answer E

Solution Solution 1

We have **water : sugar : lemon juice** = $4 \cdot 2 : 2 : 1 = 8 : 2 : 1$, so Luka needs $3 \cdot 8 = \boxed{\text{(E)} 24}$ cups.

Solution 2

Since the amount of sugar is twice the amount of lemon juice, Luka uses $3 \cdot 2 = 6$ cups of sugar.

Since the amount of water is 4 times the amount of sugar, he uses $6 \cdot 4 = \boxed{\text{(E)} 24}$ cups of water.

- 2 Four friends do yardwork for their neighbors over the weekend, earning \$15, \$20, \$25, and \$40, respectively. They decide to split their earnings equally among themselves. In total, how much will the friend who earned \$40 give to the others?

A. \$5 B. \$10 C. \$15 D. \$20 E. \$25

Answer C

Solution The friends earn $\$(15 + 20 + 25 + 40) = \$100$ in total. Since they decided to split their earnings equally, it follows that each person will get $\$\left(\frac{100}{4}\right) = \$25$. Since the friend who

earned \$40 will need to leave with \$25, he will have to give $\$(40 - 25) = \boxed{\text{(C)} \$15}$ to the others.

- 3 Carrie has a rectangular garden that measures 6 feet by 8 feet. She plants the entire garden with strawberry plants. Carrie is able to plant 4 strawberry plants per square foot, and she harvests an average of 10 strawberries per plant. How many strawberries can she expect to harvest?

A. 560 B. 960 C. 1120 D. 1920 E. 3840

Answer D

Solution Solution 1

Note that the unit of the answer is strawberries, which is the product of

- square feet
- plants per square foot
- strawberries per plant

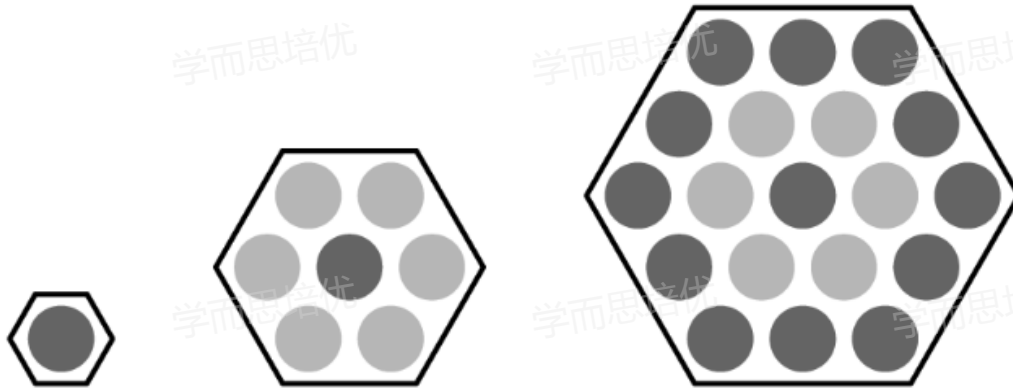
By conversion factors, we have

$$6 \cdot 8 \cdot 4 \cdot 10 \text{strawberries} = \boxed{\text{(D)} 1920} \text{strawberries.}$$

Solution 2

The area of the garden is $6 \cdot 8 = 48$ square feet. Since Carrie plants 4 strawberry plants per square foot, there are a total of $48 \cdot 4 = 192$ strawberry plants, each of which produces 10 strawberries on average. Accordingly, she can expect to harvest $192 \cdot 10 = \boxed{\text{(D)} 1920}$ strawberries.

- 4 Three hexagons of increasing size are shown below. Suppose the dot pattern continues so that each successive hexagon contains one more band of dots. How many dots are in the next hexagon?



- A. 35 B. 37 C. 39 D. 43 E. 49

Answer B

Solution **Solution 1**

Looking at the rows of each hexagon, we see that the first hexagon has 1 dot, the second has $2 + 3 + 2$ dots, and the third has $3 + 4 + 5 + 4 + 3$ dots. Given the way the hexagons are constructed, it is clear that this pattern continues. Hence, the fourth hexagon has $4 + 5 + 6 + 7 + 6 + 5 + 4 = \boxed{\text{(B) } 37}$ dots.

Solution 2

The dots in the next hexagon have four bands. From innermost to outermost:

1. The first band has 1 dot.
2. The second band has 6 dots: 1 dot at each vertex of the hexagon.
3. The third band has $6 + 6 \cdot 1 = 12$ dots: 1 dot at each vertex of the hexagon and 1 other dot on each edge of the hexagon.
4. The fourth band has $6 + 6 \cdot 2 = 18$ dots: 1 dot at each vertex of the hexagon and 2 other dots on each edge of the hexagon.

Together, the answer is $1 + 6 + 12 + 18 = \boxed{\text{(B) } 37}$.

- 5 Three fourths of a pitcher is filled with pineapple juice. The pitcher is emptied by pouring an equal amount of juice into each of 5 cups. What percent of the total capacity of the pitcher did each cup receive?

A. 5

B. 10

C. 15

D. 20

E. 25

Answer C

Solution **Solution 1**

Each cup is filled with $\frac{3}{4} \cdot \frac{1}{5} = \frac{3}{20}$ of the amount of juice in the pitcher, so the percentage is $\frac{3}{20} \cdot 100 = \boxed{\text{(C) } 15}$.

Solution 2

The pitcher is $\frac{3}{4}$ full, i.e. 75% full. Therefore each cup receives $\frac{75}{5} = \boxed{\text{(C) } 15}$ percent of the total capacity.

6

Aaron, Darren, Karen, Maren, and Sharon rode on a small train that has five cars that seat one person each. Maren sat in the last car. Aaron sat directly behind Sharon. Darren sat in one of the cars in front of Aaron. At least one person sat between Karen and Darren. Who sat in the middle car?

A. Aaron

B. Darren

C. Karen

D. Maren

E. Sharon

Answer A

Solution **Solution 1**

Write the order of the cars as $\square\square\square\square\square$, where the left end of the row represents the back of the train and the right end represents the front. Call the people *A*, *D*, *K*, *M*, and *S* respectively. The first condition gives $M\square\square\square\square$, so we try $MAS\square\square$, $M\square AS\square$, and $M\square\square AS$. In the first case, as *D* sat in front of *A*, we must have $MASDK$ or $MASKD$, which do not comply with the last condition. In the second case, we obtain $MKASD$, which works, while the third case is obviously impossible since it results in there being no way for *D* to sit in front of *A*. It follows that, with the only possible arrangement being $MKASD$, the person sitting in the middle car is $\boxed{\text{(A) } \text{Aaron}}$.

Solution 2

Follow the first few steps of Solution 1. We must have $M□□□□$, and also have $AS(\text{anything})D$ and $K \dots D$. There are only 4 spaces available for A, S, D, K , so the only possible arrangement of them is $KASD$, so the arrangement is $MKASD$, so the person in the middle car is **(A) Aaron**.

7 How many integers between 2020 and 2400 have four distinct digits arranged in increasing order? (For example, 2347 is one integer.).

- A. 9 B. 10 C. 15 D. 21 E. 28

Answer C

Solution Solution 1

Firstly, we can observe that the second digit of such a number cannot be 1 or 2 because the digits must be distinct and in increasing order. The second digit also cannot be 4 as the number must be less than 2400, so the second digit must be 3. It remains to choose the latter two digits, which must be 2 distinct digits from $\{4, 5, 6, 7, 8, 9\}$. That can be done in $\binom{6}{2} = \frac{6 \cdot 5}{2 \cdot 1} = 15$ ways; there is then only 1 way to order the digits, namely in increasing order. This means the answer is **(C) 15**.

Solution 2

As in Solution 1, we find that the first two digits must be 23, and the third digit must be at least 4 because the digits can not repeat. If it is 4, then there are 5 choices for the last digit, namely 5, 6, 7, 8, or 9. Similarly, if the third digit is 5, there are 4 choices for the last digit, namely 6, 7, 8, and 9; if 6, there are 3 choices; if 7, there are 2 choices; and if 8, there is 1 choice. It follows that the total number of such integers is $5 + 4 + 3 + 2 + 1 = \mathbf{(C) 15}$.

Ricardo has 2020 coins, some of which are pennies (1-cent coins) and the rest of which are nickels (5-cent coins). He has at least one penny and at least one nickel. What is the difference in cents between the greatest possible and least amounts of money that Ricardo can have?

- A. 806 B. 8068 C. 8072 D. 8076 E. 8082

Answer C

Solution **Solution 1**

Clearly, the amount of money Ricardo has will be maximized when he has the maximum number of nickels. Since he must have at least one penny, the greatest number of nickels he can have is 2019, giving a total of $(2019 \cdot 5 + 1)$ cents. Analogously, the amount of money he has will be least when he has the greatest number of pennies; as he must have at least one nickel, the greatest number of pennies he can have is also 2019, giving him a total of $(2019 \cdot 1 + 5)$ cents. Hence the required difference is

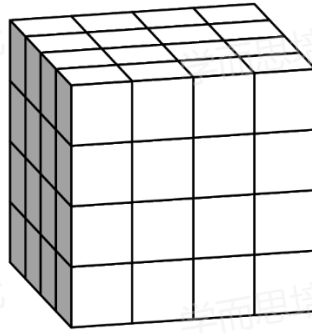
$$(2019 \cdot 5 + 1) - (2019 \cdot 1 + 5) = 2019 \cdot 4 - 4 = 4 \cdot 2018 = \boxed{\text{(C)} 8072}$$

Solution 2

Suppose Ricardo has p pennies, so then he has $(2020 - p)$ nickels. In order to have at least one penny and at least one nickel, we require $p \geq 1$ and $2020 - p \geq 1$, i.e. $1 \leq p \leq 2019$. The number of cents he has is $p + 5(2020 - p) = 10100 - 4p$, so the maximum is $10100 - 4 \cdot 1$ and the minimum is $10100 - 4 \cdot 2019$, and the difference is therefore

$$(10100 - 4 \cdot 1) - (10100 - 4 \cdot 2019) = 4 \cdot 2019 - 4 = 4 \cdot 2018 = \boxed{\text{(C)} 8072}$$

- 9 Akash's birthday cake is in the form of a $4 \times 4 \times 4$ inch cube. The cake has icing on the top and the four side faces, and no icing on the bottom. Suppose the cake is cut into 64 smaller cubes, each measuring $1 \times 1 \times 1$ inch, as shown below. How many of the small pieces will have icing on exactly two sides?



- A. 12 B. 16 C. 18 D. 20 E. 24

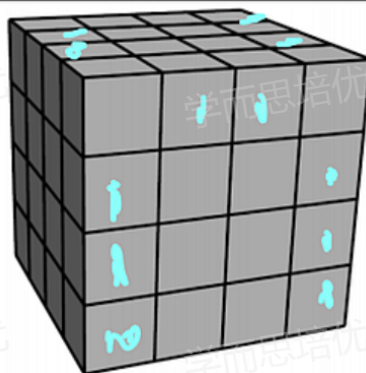
Answer D

Solution **Solution 1**

Notice that, for a small cube which does not form part of the bottom face, it will have exactly 2 faces with icing on them only if it is one of the 2 center cubes of an edge of the larger cube. There are $12 - 4 = 8$ such edges (as we exclude the 4 edges of the bottom face), so this case yields $2 \cdot 8 = 16$ small cubes. As for the bottom face, we can see that only the 4 corner cubes have exactly 2 faces with icing, so the total is $16 + 4 = \boxed{\text{(D) } 20}$.

Solution 2

The following diagram shows 12 of the small cubes having exactly 2 faces with icing on them; that is all of them except for those on the hidden face directly opposite the front face.



But the hidden face is an exact copy of the front face, so the answer is $12 + 8 = \boxed{\text{(D) } 20}$.

10

Zara has a collection of 4 marbles: an Aggie, a Bumblebee, a Stealie, and a Tiger. She wants to display them in a row on a shelf, but does not want to put the Stealie and the Tiger next to one

another. In how many ways can she do this?

A. 6

B. 8

C. 12

D. 18

E. 24

Answer C

Solution **Solution 1**

Let the Aggie, Bumblebee, Steelie, and Tiger, be referred to by A, B, S , and T , respectively.

If we ignore the constraint that S and T cannot be next to each other, we get a total of

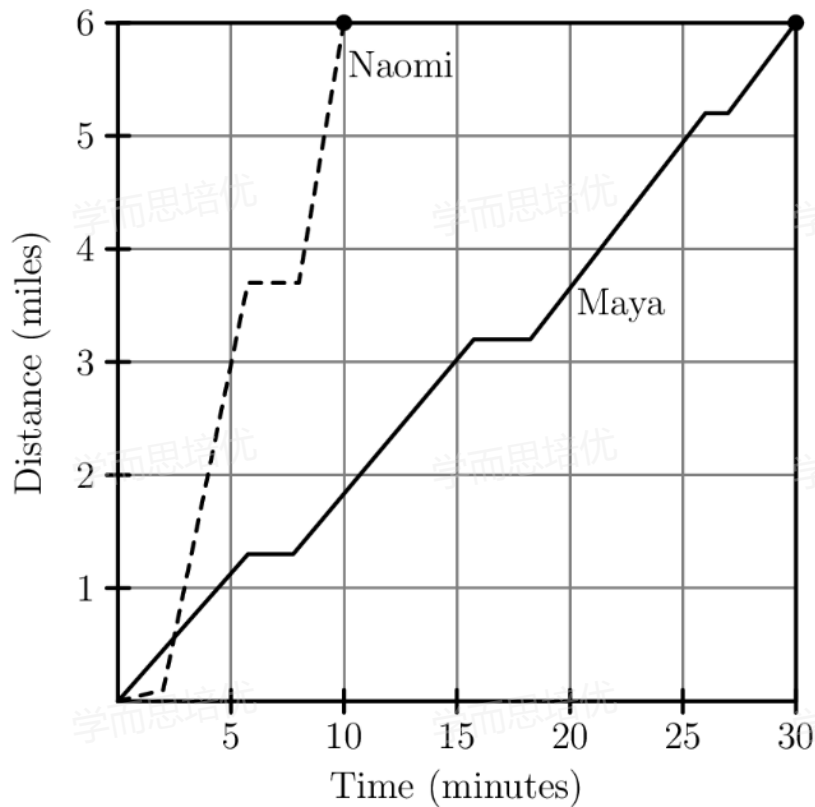
$4! = 24$ ways to arrange the 4 marbles. We now simply have to subtract out the number of ways that S and T can be next to each other. If we place S and T next to each other in that order, then there are three places that we can place them, namely in the first two slots, in the second two slots, or in the last two slots (i.e. $ST\Box\Box, \Box ST\Box, \Box\Box ST$). However, we could also have placed S and T in the opposite order (i.e. $TS\Box\Box, \Box TS\Box, \Box\Box TS$). Thus there are 6 ways of placing S and T directly next to each other. Next, notice that for each of these placements, we have two open slots for placing A and B . Specifically, we can place A in the first open slot and B in the second open slot or switch their order and place B in the first open slot and A in the second open slot. This gives us a total of $6 \times 2 = 12$ ways to place S and T next to each other. Subtracting this from the total number of arrangements gives us $24 - 12 = 12$ total arrangements $\Rightarrow$ **(C) 12**.

We can also solve this problem directly by looking at the number of ways that we can place S and T such that they are not directly next to each other. Observe that there are three ways to place S and T (in that order) into the four slots so they are not next to each other (i.e. $S\Box T\Box, \Box S\Box T, S\Box\Box T$). However, we could also have placed S and T in the opposite order (i.e. $T\Box S\Box, \Box T\Box S, T\Box\Box S$). Thus there are 6 ways of placing S and T so that they are not next to each other. Next, notice that for each of these placements, we have two open slots for placing A and B . Specifically, we can place A in the first open slot and B in the second open slot or switch their order and place B in the first open slot and A in the second open slot. This gives us a total of $6 \times 2 = 12$ ways to place S and T such that they are not next to each other $\Rightarrow$ **(C) 12**.

Solution 2

Let's try complementary counting. There $4!$ ways to arrange the 4 marbles. However, there are $2 \cdot 3!$ arrangements where Steelie and Tiger are next to each other. (Think about permutations of the element ST, A, and B or TS, A, and B). Thus, $4! - 2 \cdot 3! = \boxed{\text{(C) } 12}$

- 11 After school, Maya and Naomi headed to the beach, 6 miles away. Maya decided to bike while Naomi took a bus. The graph below shows their journeys, indicating the time and distance traveled. What was the difference, in miles per hour, between Naomi's and Maya's average speeds?



- A. 6 B. 12 C. 18 D. 20 E. 24

Answer E

Solution Naomi travels 6 miles in a time of 10 minutes, which is equivalent to $\frac{1}{6}$ of an hour. Since $\text{speed} = \frac{\text{distance}}{\text{time}}$, her speed is $\frac{6}{(\frac{1}{6})} = 36$ mph. By a similar calculation, Maya's speed is 12 mph, so the answer is $36 - 12 = \boxed{\text{(E) } 24}$.

12 For a positive integer n , the factorial notation $n!$ represents the product of the integers from n to

1. What value of N satisfies the following equation?

$$5! \cdot 9! = 12 \cdot N!$$

A. 10

B. 11

C. 12

D. 13

E. 14

Answer A

Solution We have $5! = 2 \cdot 3 \cdot 4 \cdot 5$, and $2 \cdot 5 \cdot 9! = 10 \cdot 9! = 10!$. Therefore, the equation becomes

$$3 \cdot 4 \cdot 10! = 12 \cdot N!, \text{ and so } 12 \cdot 10! = 12 \cdot N!. \text{ Cancelling the 12s, it is clear that } N = \boxed{\text{(A) } 10}$$

13 Jamal has a drawer containing 6 green socks, 18 purple socks, and 12 orange socks. After adding more purple socks, Jamal noticed that there is now a 60% chance that a sock randomly selected from the drawer is purple. How many purple socks did Jamal add?

A. 6

B. 9

C. 12

D. 18

E. 24

Answer B

Solution **Solution 1**

After Jamal adds x purple socks, he has $(18 + x)$ purple socks and

$6 + 18 + 12 + x = (36 + x)$ total socks. This means the probability of drawing a purple sock

is $\frac{18 + x}{36 + x}$, so we obtain

$$\frac{18 + x}{36 + x} = \frac{3}{5}$$

Since $\frac{18 + 9}{36 + 9} = \frac{27}{45} = \frac{3}{5}$, the answer is $\boxed{\text{(B) } 9}$.

Solution 2

6 green socks and 12 orange socks together should be $100\% - 60\% = 40\%$ of the new total

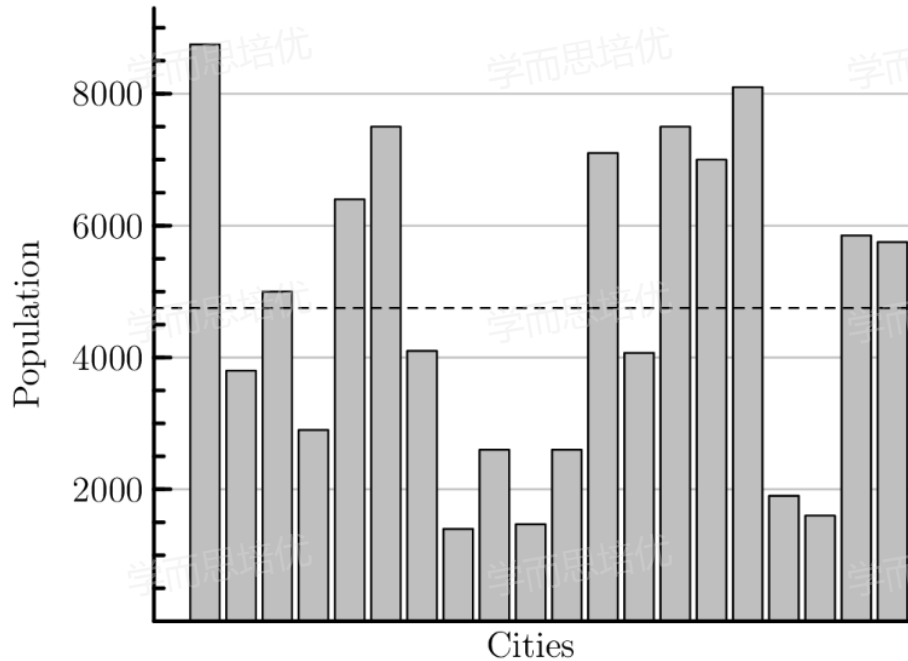
number of socks, so that new total must be $\frac{6 + 12}{0.4} = 45$. Therefore,

$$45 - 6 - 18 - 12 = \boxed{\text{(B) } 9} \text{ purple socks were added.}$$

14

There are 20 cities in the County of Newton. Their populations are shown in the bar chart below.

The average population of all the cities is indicated by the horizontal dashed line. Which of the following is closest to the total population of all 20 cities?



A. 65,000

B. 75,000

C. 85,000

D. 95,000

E. 105,000

Answer D

Solution **Solution 1**

We can see that the dotted line is exactly halfway between 4,500 and 5,000, so it is at 4,750.

As this is the average population of all 20 cities, the total population is simply

$$4,750 \cdot 20 = \boxed{\text{(D) } 95,000}.$$

Solution 2

The dotted line is slightly below 5000. Since $5000 \cdot 20 = 100,000$, the answer is slightly below

this, so $\boxed{\text{(D) } 95,000}$ would be the best choice to guess.

15

Suppose 15% of x equals 20% of y . What percentage of x is y ?

A. 5

B. 35

C. 75

D. $133\frac{1}{3}$

E. 300

Answer C

Solution **Solution 1**

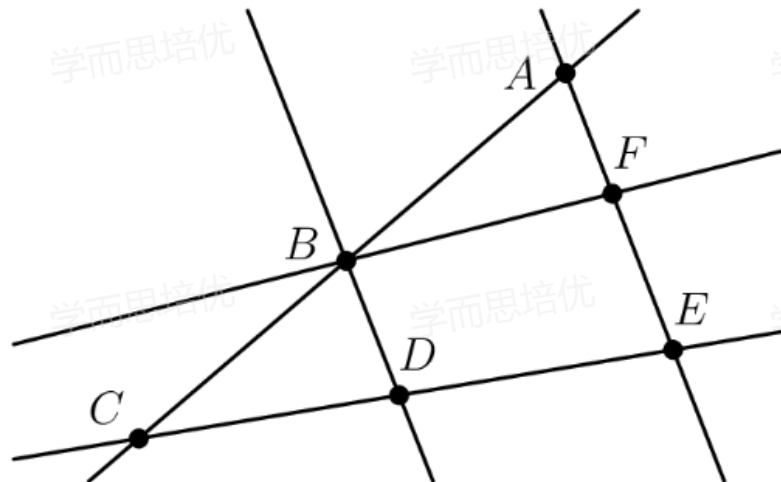
Since $20\% = \frac{1}{5}$, multiplying the given condition by 5 shows that y is $15 \cdot 5 = \boxed{\text{(C) } 75}$ percent of x .

Solution 2

Letting $x = 100$ (without loss of generality), the condition becomes

$0.15 \cdot 100 = 0.2 \cdot y \Rightarrow 15 = \frac{y}{5} \Rightarrow y = 75$. Clearly, it follows that y is 75% of x , so the answer is $\boxed{\text{(C) } 75}$.

- 16 Each of the points A, B, C, D, E , and F in the figure below represents a different digit from 1 to 6. Each of the five lines shown passes through some of these points. The digits along each line are added to produce five sums, one for each line. The total of the five sums is 47. What is the digit represented by B ?



A. 1

B. 2

C. 3

D. 4

E. 5

Answer E

Solution **Solution 1**

We can form the following expressions for the sum along each line:

$$\begin{cases} A + B + C \\ A + E + F \\ C + D + E \\ B + D \\ B + F \end{cases}$$

Adding these together, we must have $2A + 3B + 2C + 2D + 2E + 2F = 47$, i.e.

$2(A + B + C + D + E + F) + B = 47$. Since A, B, C, D, E, F are unique integers between 1 and 6, we obtain $A + B + C + D + E + F = 1 + 2 + 3 + 4 + 5 + 6 = 21$ (where the order doesn't matter as addition is commutative), so our equation simplifies to $42 + B = 47$. This means $B = \boxed{\text{(E)} 5}$.

Solution 2

Following the first few steps of Solution 1, we have $2(A + C + D + E + F) + 3B = 47$.

Because an even number $2(A + C + D + E + F)$ subtracted from an odd number (47) is always odd, we know that $3B$ is odd, showing that B is odd. Now we know that B is 1, 3, or 5. If we try $B = 1$, we get $43 = 47$, which is false. Testing $B = 3$, we get $45 = 47$, which is also false. Therefore, we have $B = \boxed{\text{(E)} 5}$.

17 How many positive integer factors of 2020 have more than 3 factors? (As an example, 12 has 6 factors, namely 1, 2, 3, 4, 6, and 12.)

- A. 6 B. 7 C. 8 D. 9 E. 10

Answer B

Solution Solution 1

Since $2020 = 2^2 \cdot 5 \cdot 101$, we can simply list its factors:

1, 2, 4, 5, 10, 20, 101, 202, 404, 505, 1010, 2020.

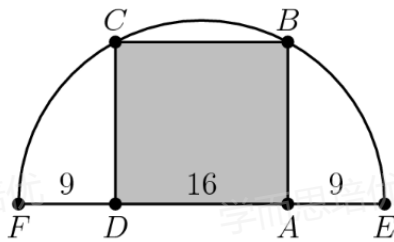
There are 12 factors; only 1, 2, 4, 5, 101 don't have over 3 factors, so the remaining

$12 - 5 = \boxed{\text{(B)} 7}$ factors have more than 3 factors.

Solution 2

As in Solution 1, we prime factorize **2020** as $2^2 \cdot 5 \cdot 101$, and we recall the standard formula that the number of positive factors of an integer is found by adding 1 to each exponent in its prime factorization, and then multiplying these. Thus **2020** has $(2 + 1)(1 + 1)(1 + 1) = 12$ factors. The only number which has one factor is 1. For a number to have exactly two factors, it must be prime, and the only prime factors of **2020** are 2, 5, and 101. For a number to have three factors, it must be a square of a prime (this follows from the standard formula mentioned above), and from the prime factorization, the only square of a prime that is a factor of **2020** is 4. Thus, there are 5 factors of **2020** which themselves have 1, 2, or 3 factors (namely 1, 2, 4, 5, and 101), so the number of factors of **2020** that have more than 3 factors is $12 - 5 = \boxed{\text{(B)} 7}$.

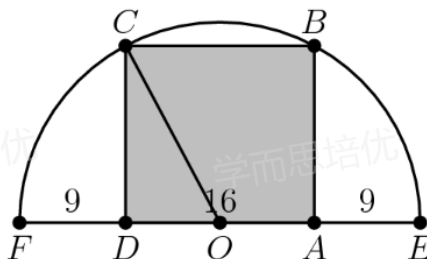
- 18 Rectangle $ABCD$ is inscribed in a semicircle with diameter $\overline{FE}$, as shown in the figure. Let $DA = 16$, and let $FD = AE = 9$. What is the area of $ABCD$?



- A. 240 B. 248 C. 256 D. 264 E. 272

Answer A

Solution **Solution 1**



Let O be the center of the semicircle. The diameter of the semicircle is $9 + 16 + 9 = 34$, so $OC = 17$. By symmetry, O is the midpoint of DA , so $OD = OA = \frac{16}{2} = 8$. By the Pythagorean theorem in right-angled triangle ODC (or OBA), we have that CD (or AB) is

$\sqrt{17^2 - 8^2} = 15$. Accordingly, the area of $ABCD$ is $16 \cdot 15 = \boxed{\text{(A) } 240}$.

Solution 2

Let the midpoint of segment FE be the origin. Evidently, point $D = (-8, 0)$ and $A = (8, 0)$.

Since points C and B share x -coordinates with D and A respectively, it suffices to find the y -coordinate of B (which will be the height of the rectangle) and multiply this by DA (which

we know is 16). The radius of the semicircle is $\frac{9 + 16 + 9}{2} = 17$, so the whole circle has

equation $x^2 + y^2 = 289$; as already stated, B has the same x -coordinate as A , i.e. 8, so

substituting this into the equation shows that $y = \pm 15$. Since $y > 0$ at B , the y -coordinate of

B is 15. Therefore, the answer is $16 \cdot 15 = \boxed{\text{(A) } 240}$.

- 19 A number is called flippy if its digits alternate between two distinct digits. For example, 2020 and 37373 are flippy, but 3883 and 123123 are not. How many five-digit flippy numbers are divisible by 15?

A. 3 B. 4 C. 5 D. 6 E. 8

Answer B

Solution Solution 1

A number is divisible by 15 precisely if it is divisible by 3 and 5. The latter means the last digit must be either 5 or 0, and the former means the sum of the digits must be divisible by 3.

If the last digit is 0, the first digit would be 0 (because the digits alternate), which is not possible. Hence the last digit must be 5, and the number is of the form $5\square 5\square 5$. If the

unknown digit is x , we deduce $5 + x + 5 + x + 5 \equiv 0 \pmod{3} \Rightarrow 2x \equiv 0 \pmod{3}$. We know 2^{-1} exists modulo 3 because 2 is relatively prime to 3, so we conclude that x (i.e. the second and fourth digit of the number) must be a multiple of 3. It can be 0, 3, 6, or 9, so

there are $\boxed{\text{(B) } 4}$ options: 50505, 53535, 56565, and 59595.

Solution 2

After finding out that the last digit must be 5, the number is of the form $5\square5\square5$. If the unknown digit is x , we can find that one of the solutions to x is 0, since $5 + 5 + 5$ is equal to 15, which is divisible by 3. After trying every one digit number, you'll notice that x must be a multiple of 3, meaning that $x = 0, 3, 6$, or 9. 50505, 53535, 56565, and 59595 are the **(B) 4** solutions to this question.

- 20 A scientist walking through a forest recorded as integers the heights of 5 trees standing in a row. She observed that each tree was either twice as tall or half as tall as the one to its right. Unfortunately some of her data was lost when rain fell on her notebook. Her notes are shown below, with blanks indicating the missing numbers. Based on her observations, the scientist was able to reconstruct the lost data. What was the average height of the trees, in meters?

Tree 1	— meters
Tree 2	11 meters
Tree 3	— meters
Tree 4	— meters
Tree 5	— meters
Average height	— .2 meters

- A. 22.2 B. 24.2 C. 33.2 D. 35.2 E. 37.2

Answer B

Solution **Solution 1**

We will show that 22, 11, 22, 44, and 22 meters are the heights of the trees from left to right.

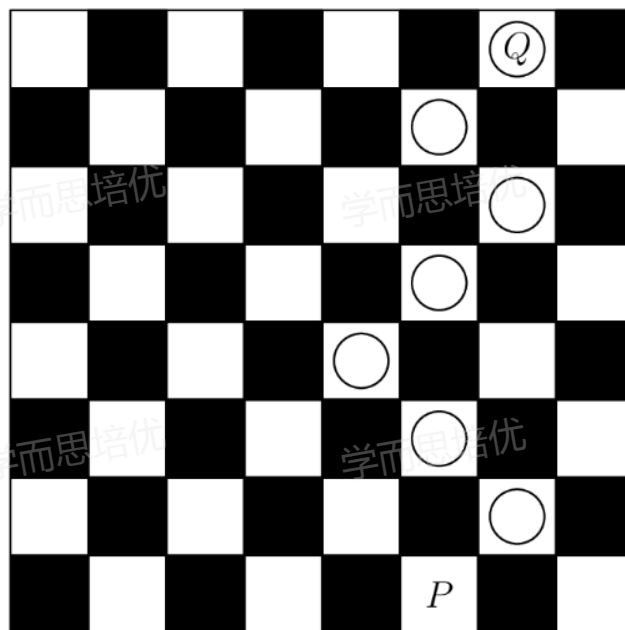
We are given that all tree heights are integers, so since Tree 2 has height 11 meters, we can deduce that Trees 1 and 3 both have a height of 22 meters. There are now three possible cases for the heights of Trees 4 and 5 (in order for them to be integers), namely heights of 11 and 22, 44 and 88, or 44 and 22. Checking each of these, in the first case, the average is 17.6 meters, which doesn't end in .2 as the problem requires. Therefore, we consider the other cases. With 44 and 88, the average is 37.4 meters, which again does not

end in .2, but with **44** and **22**, the average is **24.2** meters, which does. Consequently, the answer is **(B) 24.2**.

Solution 2

Notice the average height of the trees ends with **0.2**; therefore, the sum of all five heights of the trees must end with **1** or **6**. ($0.2 \cdot 5 = 1$) We already know Tree 2 is **11** meters tall. Both Tree 1 and Tree 3 must **22** meters tall - since neither can be **5.5**. Once again, apply our observation for solving for the Tree 4's height. Tree 4 can't be **11** meters for the sum of the five tree heights to still end with **1**. Therefore, the Tree 4 is **44** meters tall. Now, Tree 5 can either be **22** or **88**. Find the average height for both cases of Tree 5. Doing this, we realize the Tree 5 must be **22** for the average height to end with **0.2** and that the average height is **(B) 24.2**.

- 21 A game board consists of **64** squares that alternate in color between black and white. The figure below shows square **P** in the bottom row and square **Q** in the top row. A marker is placed at **P**. A step consists of moving the marker onto one of the adjoining white squares in the row above. How many **7**-step paths are there from **P** to **Q**? (The figure shows a sample path.)



A. 28

B. 30

C. 32

D. 33

E. 35

Answer A

Solution 1

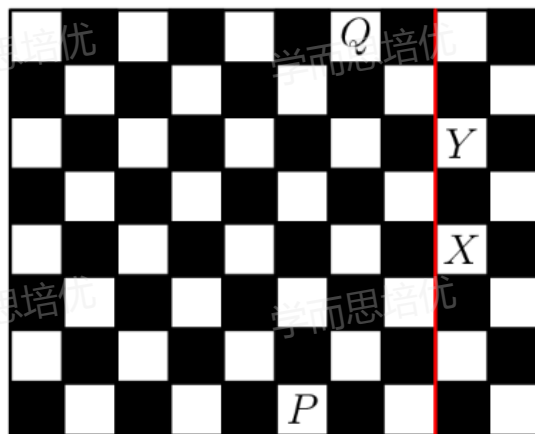
Notice that, to step onto any particular white square, the marker must have come from one of the 1 or 2 white squares immediately beneath it (since the marker can only move on white squares). This means that the number of ways to move from P to that square is the sum of the number of ways to move from P to each of the white squares immediately beneath it (also called the Waterfall Method). To solve the problem, we can accordingly construct the following diagram, where each number in a square is calculated as the sum of the numbers on the white squares immediately beneath that square (and thus will represent the number of ways to remove from P to that square, as already stated).

						28	
					19	9	
				10	9		
			4		6		3
		1		3		3	
			1		2		1
				1		1	
					1		

The answer is therefore **(A) 28**.

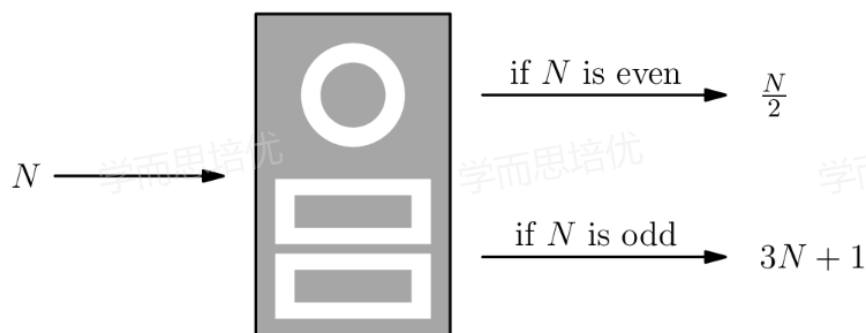
Solution 2

Suppose we "extend" the chessboard infinitely with 2 additional columns to the right, as shown below. The red line shows the right-hand edge of the original board.



The total number of paths from P to Q , including invalid paths which cross over the red line, is then the number of paths which make 4 steps up-and-right and 3 steps up-and-left, which is $\binom{4+3}{3} = \binom{7}{3} = 35$. We need to subtract the number of invalid paths, i.e. the number of paths that pass through X or Y . To get to X , the marker has to make 3 up-and-right steps, after which it can proceed to Q with 3 steps up-and-left and 1 step up-and-right. Thus, the number of paths from P to Q that pass through X is $1 \cdot \binom{3+1}{3} = 4$. Similarly, the number of paths that pass through Y is $\binom{4+1}{1} \cdot 1 = 5$. However, we have now double-counted the invalid paths which pass through both X and Y ; from the diagram, it is clear that there are only 2 of these (as the marker can get from X to Y by a step up-and-left and a step-up-and-right in either order). Hence the number of invalid paths is $4 + 5 - 2 = 7$, and the number of valid paths from P to Q is $35 - 7 = \boxed{\text{(A) } 28}$.

- 22 When a positive integer N is fed into a machine, the output is a number calculated according to the rule shown below.



For example, starting with an input of $N = 7$, the machine will output $3 \cdot 7 + 1 = 22$. Then if the output is repeatedly inserted into the machine five more times, the final output is 26.

$$7 \rightarrow 22 \rightarrow 11 \rightarrow 34 \rightarrow 17 \rightarrow 52 \rightarrow 26$$

When the same 6-step process is applied to a different starting value of N , the final output is 1.

What is the sum of all such integers N ?

$$N \rightarrow \_ \rightarrow \_ \rightarrow \_ \rightarrow \_ \rightarrow \_ \rightarrow 1$$

A. 73

B. 74

C. 75

D. 82

E. 83

Answer E

Solution We start with final output of 1 and work backward, taking care to consider all possible inputs that could have resulted in any particular output. This produces following set of possibilities each stage:

$$\{1\} \rightarrow \{2\} \rightarrow \{4\} \rightarrow \{1, 8\} \rightarrow \{2, 16\} \rightarrow \{4, 5, 32\} \rightarrow \{1, 8, 10, 64\}$$

where, for example, 2 must come from 4 (as there is no integer n satisfying $3n + 1 = 2$), but 16 could come from 32 or 5 (as $\frac{32}{2} = 3 \cdot 5 + 1 = 16$, and 32 is even while 5 is odd). By

construction, last set in this sequence contains all the numbers which will lead to number 1 to end of the 6-step process, and sum is $1 + 8 + 10 + 64 = \boxed{\text{(E)} 83}$.

23 Five different awards are to be given to three students. Each student will receive at least one award. In how many different ways can the awards be distributed?

A. 120

B. 150

C. 180

D. 210

E. 240

Answer B

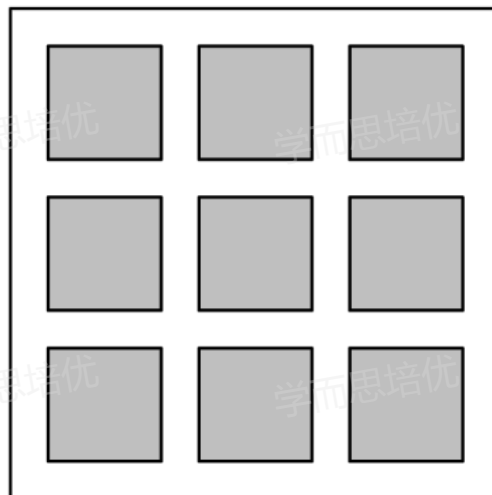
Solution Firstly, observe that a single student can't receive 4 or 5 awards because this would mean that one of the other students receives no awards. Thus, each student must receive either 1, 2, or 3 awards. If a student receives 3 awards, then the other two students must each receive 1 award; if a student receives 2 awards, then another student must also receive 2 awards and the remaining student must receive 1 award. We consider each of these two cases in turn.

If a student receives three awards, there are **3** ways to choose which student this is, and $\binom{5}{3}$ ways to give that student **3** out of the **5** awards. Next, there are **2** students left and **2** awards to give out, with each student getting one award. There are clearly just **2** ways to distribute these two awards out, giving $3 \cdot \binom{5}{3} \cdot 2 = 60$ ways to distribute the awards in this case.

In the other case, two students receive **2** awards and one student receives **1** award. We know there are **3** choices for which student gets **1** award. There are $\binom{3}{1}$ ways to do this.

Then, there are $\binom{5}{2}$ ways to give the first student his two awards, leaving **3** awards yet to be distributed. There are then $\binom{3}{2}$ ways to give the second student his **2** awards. Finally, there is only **1** student and **1** award left, so there is only **1** way to distribute this award. This results in $\binom{5}{2} \cdot \binom{3}{2} \cdot 1 \cdot 3 = 90$ ways to distribute the awards in this case. Adding the results of these two cases, we get $60 + 90 = \boxed{\text{(B) } 150}$.

- 24 A large square region is paved with n^2 gray square tiles, each measuring s inches on a side. A border d inches wide surrounds each tile. The figure below shows the case for $n = 3$. When $n = 24$, the 576 gray tiles cover 64% of the area of the large square region. What is the ratio $\frac{d}{s}$ for this larger value of n ?



- A. $\frac{6}{25}$ B. $\frac{1}{4}$ C. $\frac{9}{25}$ D. $\frac{7}{16}$
 E. $\frac{9}{16}$

Answer A

Solution 1

The area of the shaded region is $(24s)^2$. To find the area of the large square, we note that there is a d -inch border between each of the 23 pairs of consecutive squares, as well as from between the first/last squares and the large square, for a total of $23 + 2 = 25$ times the length of the border, i.e. $25d$. Adding this to the total length of the consecutive squares, which is $24s$, the side length of the large square is $(24s + 25d)$, yielding the equation

$$\frac{(24s)^2}{(24s + 25d)^2} = \frac{64}{100}. \text{ Taking the square root of both sides (and using the fact that lengths are non-negative) gives } \frac{24s}{24s + 25d} = \frac{8}{10} = \frac{4}{5}, \text{ and cross-multiplying now gives}$$

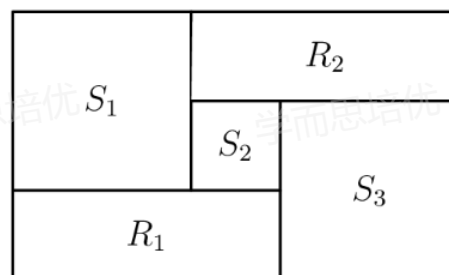
$$120s = 96s + 100d \Rightarrow 24s = 100d \Rightarrow \frac{d}{s} = \frac{24}{100} = \boxed{\text{(A)} \frac{6}{25}}.$$

Solution 2

WLOG (Without Loss of Generality), we may let $s = 1$ (since d will be determined by the scale of s , and we are only interested in the ratio $\frac{d}{s}$). Then, as the total area of the 576 gray tiles is simply 576, the large square has area $\frac{576}{0.64} = 900$, making the side of the large square $\sqrt{900} = 30$. As in Solution 1, the side length of the large square consists of the total length of the gray tiles and 25 lots of the border, so the length of the border is

$$d = \frac{30 - 24}{25} = \frac{6}{25}. \text{ Since } \frac{d}{s} = d \text{ if } s = 1, \text{ the answer is } \boxed{\text{(A)} \frac{6}{25}}.$$

- 25 Rectangles R_1 and R_2 , and squares S_1 , S_2 , and S_3 , shown below, combine to form a rectangle that is 3322 units wide and 2020 units high. What is the side length of S_2 in units?



- A. 651 B. 655 C. 656 D. 662 E. 666

Answer A

Solution Solution 1

Let the side length of each square S_k be s_k . Then, from the diagram, we can line up the top horizontal lengths of S_1 , S_2 , and S_3 to cover the top side of the large rectangle, so $s_1 + s_2 + s_3 = 3322$. Similarly, the short side of R_2 will be $s_1 - s_2$, and lining this up with the left side of S_3 to cover the vertical side of the large rectangle gives $s_1 - s_2 + s_3 = 2020$. We subtract the second equation from the first to obtain $2s_2 = 1302$, and thus $s_2 = \boxed{\text{(A) } 651}$.

Solution 2

Assuming that the problem is well-posed, it should be true in the particular case where $S_1 \cong S_3$ and $R_1 \cong R_2$. Let the sum of the side lengths of S_1 and S_3 be x , and let the length of square S_2 be y . We then have the system

$$\begin{cases} x + y = 3322 \\ x - y = 2020 \end{cases}$$

which we solve to determine $y = \boxed{\text{(A) } 651}$.