

2017 AMC 10B

- 1 Mary thought of a positive two-digit number. She multiplied it by 3 and added 11. Then she switched the digits of the result, obtaining a number between 71 and 75, inclusive. What was Mary's number? (2017 AMC 10B Problem, Question#1)

A. 11 B. 12 C. 13 D. 14 E. 15

Answer B

Solution Let her 2-digit number be x . Multiplying by 3 makes it a multiple of 3, meaning that the sum of its digits is divisible by 3. Adding on 11 increases the sum of the digits by $1 + 1 = 2$, and reversing the digits keeps the sum of the digits the same; this means that the resulting number must be 2 more than a multiple of 3. There are two such numbers between 71 and 75 : 71: and 74 .Now that we have narrowed down the choices, we can simply test the answers to see which one will provide a two-digit number when the steps are reversed:For 71,we reverse the digits, resulting in 17 Subtracting 11, we get 6.We can already see that dividing this by 3 will not be a two-digit number, so 71 does not meet our requirements.Therefore, the answer must be the reversed steps applied to 74.We have the following: $74 \rightarrow 47 \rightarrow 36 \rightarrow 12$ Therefore, our answer is 12.

Working backwards, we reverse the digits of each number from 71 ~ 75 and subtract 11 from each, so we have 6,16,26,36,46The only numbers from this list that are divisible by 3 are 6 and 36. We divide both by 3, yielding 2 and 12. Since 2 is not a two-digit number, the answer is 12.

You can just plug in the numbers to see which one works. When you get to 12, you multiply by 3 and add 11 to get 47. When you reverse the digits of 47, you get 74, which is within the given range. Thus, the answer is 12.

- 2 Sofia ran 5 laps around the 400-meter track at her school. For each lap, she ran the first 100 meters at an average speed of 4 meters per second and the remaining 300 meters at an average speed of 5 meters per second. How much time did Sofia take running the 5 laps? (2017 AMC 10B Problem, Question#2)
- A. 5 minutes and 35 seconds B. 6 minutes and 40 seconds
 C. 7 minutes and 5 seconds D. 7 minutes and 25 seconds
 E. 8 minutes and 10 seconds

Answer C

Solution If Sofia ran the first 100 meters of each lap at 4 meters per second and the remaining 300 meters of each lap at 5 meters per second, then she took

$$\frac{100}{4} + \frac{300}{5} = 25 + 60 = 85 \text{ seconds for each lap. Because she ran 5 laps, she took a total of } 5 \cdot 85 = 425 \text{ seconds, or 7 minutes and 5 seconds.}$$

The answer is C.

- 3 Real numbers $x, y,$ and z satisfy the inequalities $0 < x < 1, -1 < y < 0$, and $1 < z < 2$. Which of the following numbers is necessarily positive? (2017 AMC 10B Problem, Question#3)
- A. $y + x^2$ B. $y + xz$ C. $y + y^2$ D. $y + 2y^2$ E. $y + z$

Answer E

Solution Notice that $y + z$ must be positive because $|z| > |y|$. Therefore the answer is E.

The other choices:

- (A) As x grows closer to 0, x^2 decreases and thus becomes less than y .
 (B) x can be as small as possible ($x > 0$), so xz grows close to 0 as x approaches 0.
 (C) For all $-1 < y < 0$, $y > y^2$, and thus it is always negative.
 (D) The same logic as above, but when $-\frac{1}{2} < y < 0$ this time.

- 4 Suppose that x and y are nonzero real numbers such that $\frac{3x+y}{x-3y} = -2$. What is the value of $\frac{x+3y}{3x-y}$? (2017 AMC 10B Problem, Question#4)

A. -3 B. -1 C. 1 D. 2 E. 3

Answer D

Solution Rearranging, we find $3x + y = -2x + 6y$, or $5x = 5y \Rightarrow x = y$. Substituting, we can

convert the second equation into $\frac{x+3x}{3x-x} = \frac{4x}{2x} = 2$.

Substituting each x and y with 1, we see that the given equation holds true, as

$$\frac{3(1)+1}{1-3(1)} = -2. \text{ Thus, } \frac{x+3x}{3x-x} = -2.$$

Let $y = ax$. The first equation converts into $\frac{(3+a)x}{(1-3a)x} = -2$, which simplifies to

$3+a = -2(1-3a)$. After a bit of algebra we found out $a = 1$, which means that $x = y$.

Substituting $y = x$ into the second equation it becomes $\frac{4x}{2x} = 2$.

- 5 Camilla had twice as many blueberry jelly beans as cherry jelly beans. After eating 10 pieces of each kind, she now has three times as many blueberry jelly beans as cherry jelly beans. How many blueberry jelly beans did she originally have? (2017 AMC 10B Problem, Question#5)

A. 10 B. 20 C. 30 D. 40 E. 50

Answer D

Solution Denote the number of blueberry and cherry jelly beans as b and c respectively. Then

$b = 2c$ and $b - 10 = 3(c - 10)$. Substituting, we have $2c - 10 = 3c - 30$, so $c = 20$, $b = 40$.

From the problem, we see that 10 less than one of the answer choices must be a multiple of 3 and positive. The only answer choice satisfying this is 40. We can check that 40 blueberry and 20 cherry jelly beans indeed does work.

6

What is the largest number of solid $2\text{-in} \times 2\text{-in} \times 1\text{-in}$ blocks that can fit in a $3\text{-in} \times 2\text{-in} \times 3\text{-in}$ box? (2017 AMC 10B Problem, Question#6)

- A. 3 B. 4 C. 5 D. 6 E. 7

Answer B

Solution We find that the volume of the larger block is 18, and the volume of the smaller block is 4. Dividing the two, we see that only a maximum of four 2 by 2 by 1 blocks can fit inside the 3 by 3 by 2 block. Drawing it out, we see that such a configuration is indeed possible. Therefore, the answer is 4.

- 7 Samia set off on her bicycle to visit her friend, traveling at an average speed of 17 kilometers per hour. When she had gone half the distance to her friend's house, a tire went flat, and she walked the rest of the way at 5 kilometers per hour. In all it took her 44 minutes to reach her friend's house. In kilometers rounded to the nearest tenth, how far did Samia walk? (2017 AMC 10B Problem, Question#7)

- A. 2.0 B. 2.2 C. 2.8 D. 3.4 E. 4.4

Answer C

Solution Let's call the distance that Samia had to travel in total as $2x$, so that we can avoid fractions. We know that the length of the bike ride and how far she walked are equal, so they are both $\frac{2x}{2}$, or x . She bikes at a rate of 17 kph, so she travels the distance she bikes in $\frac{7}{17}$ hours. She walks at a rate of 5 kph, so she travels the distance she walks in $\frac{x}{5}$ hours. The total time is $\frac{x}{17} + \frac{x}{5} = \frac{22x}{85}$. This is equal to $\frac{44}{60} = \frac{11}{15}$ of an hour. Solving for x , we have: $\frac{22x}{85} = \frac{11}{15}$, $\frac{2x}{85} = \frac{1}{15}$, $30x = 85$, $6x = 17$, $x = \frac{17}{6}$. Since x is the distance of how far Samia traveled by both walking and biking, and we want to know how far Samia walked to the nearest tenth, we have that Samia walked about 2.8.

Notice that Samia walks $\frac{17}{5}$ times slower than she bikes, and that she walks and bikes the same distance. Thus, the fraction of the total time that Samia will be

walking is $\frac{\frac{17}{5}}{\frac{17}{5} + \frac{5}{5}} = \frac{17}{22}$. Then, multiply this by the time $\frac{17}{22} \cdot 44 = 34$ minutes, 34

minutes is a little greater than $\frac{1}{2}$ of an hour so Samia traveled $\sim \frac{1}{2} \cdot 5 = 2.5$

kilometers. The answer choice a little greater than 2.5 is 2.8. (Note that we could've

multiplied $\frac{34}{60} = \frac{17}{30}$ by 5 and gotten the exact answer as well)

Since the distance that both rates travel are equivalent, we can find the harmonic mean of the two rates to find the average speed. Therefore, let $a = 17$ and $b = 5$,

$\bar{s} = \frac{2ab}{a+b} \Rightarrow \bar{s} = \frac{2 \cdot 17 \cdot 5}{17+5} = \frac{85}{11}$. Now, we can find the overall distance by multiply the

average speed by the time(hours), 44 minutes. Let $s = \frac{11}{15}$, $d = \bar{s} \cdot t \Rightarrow d = \frac{85}{11} \cdot \frac{11}{15} = \frac{17}{3}$

, Because the distance Samia walks is half the total distance, the distance she walks is

$\frac{17}{6}$. With some knowledge of sixths, one finds that Samia walked, in kilometers,

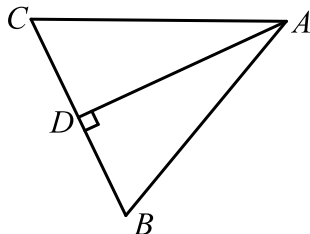
$2.8\bar{3} \approx 2.8 \Rightarrow C$.

- 8 Points $A(11, 9)$ and $B(2, -3)$ are vertices of $\triangle ABC$ with $AB = AC$. The altitude from A meets the opposite side at $D(-1, 3)$. What are the coordinates of point C ? (2017 AMC 10B Problem, Question#8)

- A. $(-8, 9)$ B. $(-4, 8)$ C. $(-4, 9)$ D. $(-2, 3)$ E. $(-1, 0)$

Answer C

Solution



Since $AB = AC$, then $\triangle ABC$ is isosceles, so $BD = CD$. Therefore, the coordinates of C are $(-1 - 3, 3 + 6) = (-4, 9)$.

Calculating the equation of the line running between points B and D , $y = -2x + 1$.

The only coordinate of C that is also on this line is $(-4, 9)$.

- 9 A radio program has a quiz consisting of 3 multiple-choice questions, each with 3 choices. A contestant wins if he or she gets 2 or more of the questions right. The contestant answers randomly to each question. What is the probability of winning?

(2017 AMC 10B Problem, Question#9)

- A. $\frac{1}{27}$ B. $\frac{1}{9}$ C. $\frac{2}{9}$ D. $\frac{7}{27}$ E. $\frac{1}{2}$

Answer D

Solution There are two ways the contestant can win.

Case 1: The contestant guesses all three right. This can only happen $\frac{1}{3} * \frac{1}{3} * \frac{1}{3} = \frac{1}{27}$ of the time.

Case 2: The contestant guesses only two right. We pick one of the questions to get wrong, 3, and this can happen $\frac{1}{3} * \frac{1}{3} * \frac{2}{3}$ of the time. Thus, $\frac{2}{27} * 3 = \frac{6}{27}$. So, in total the two cases combined equals $\frac{1}{27} + \frac{6}{27} = (\text{D}) \frac{7}{27}$.

Complementary counting is good for solving the problem and checking work if you solved it using the method above.

There are two ways the contestant can lose.

Case 1: The contestant guesses zero questions correctly.

The probability of guessing incorrectly for each question is $\frac{2}{3}$. Thus, the probability of guessing all questions incorrectly is $\frac{2}{3} * \frac{2}{3} * \frac{2}{3} = \frac{8}{27}$.

Case 2: The contestant guesses one question correctly. There are 3 ways the

contestant can guess one question correctly since there are 3 questions. The probability of guessing correctly is $\frac{1}{3}$. so the probability of guessing one correctly and two incorrectly is $3 * \frac{1}{3} * \frac{2}{3} * \frac{2}{3} = \frac{4}{9}$.

The sum of the two cases is $\frac{8}{27} + \frac{4}{9} = \frac{20}{27}$. This is the complement of what we want to the answer is $1 - \frac{20}{27} = (\text{D}) \frac{7}{27}$.

- 10 The lines with equations $ax - 2y = c$ and $2x + by = -c$ are perpendicular and intersect at $(1, -5)$. What is c ? (2017 AMC 10B Problem, Question#10)

A. -13 B. -8 C. 2 D. 8 E. 13

Answer E

Solution Writing each equation in slope–intercept form, we get $y = \frac{a}{2}x - \frac{1}{2}c$ and $y = -\frac{2}{b}x - \frac{c}{b}$. We observe the slope of each equation is $\frac{a}{2}$ and $-\frac{2}{b}$, respectively. Because the slope of a line perpendicular to a line with slope m is $-\frac{1}{m}$, we see that $\frac{a}{2} = -\frac{1}{-\frac{2}{b}}$, because it is given that the two lines are perpendicular. This equation simplifies to $a = b$. Because $(1, -5)$ is a solution of both equations, we deduce $a \times 1 - 2 \times (-5) = c$ and $2 \times 1 + b \times (-5) = -c$. Because we know that $a = b$, the equations reduce to $a + 10 = c$ and $2 - 5a = -c$. Solving this system of equations, we get $c = (\text{E})13$.

- 11 At Typico High School, 60% of the students like dancing, and the rest dislike it. Of those who like dancing, 80% say that they like it, and the rest say that they dislike it. Of those who dislike dancing, 90% say that they dislike it, and the rest say that they like it. What fraction of students who say they dislike dancing actually like it? (2017 AMC 10B Problem, Question#11)

A. 10% B. 12% C. 20% D. 25% E. $33\frac{1}{3}\%$

Answer D

Solution $60\% \cdot 20\% = 12\%$ of the people that claim that they dislike dancing actually like it, and $40\% \cdot 90\% = 36\%$. Therefore, the answer is $\frac{12\%}{12\% + 36\%} = (\text{D})25\%$.

12

Elmer's new car gives 50% better fuel efficiency. However, the new car uses diesel fuel, which is 20% more expensive per liter than the gasoline the old car used. By what percent will Elmer save money if he uses his new car instead of his old car for a long trip? (2017 AMC 10B Problem, Question#12)

- A. 20% B. $26\frac{2}{3}\%$ C. $27\frac{7}{9}\%$ D. $33\frac{1}{3}\%$ E. $66\frac{2}{3}\%$

Answer A

Solution Suppose that his old car runs at x km per liter. Then his new car runs at $\frac{3}{2}x$ km per liter, or x km per $\frac{2}{3}$ of a liter. Let the cost of the old car's fuel be c , so the trip in the old car takes xc dollars, while the trip in the new car takes $\frac{2}{3} \cdot \frac{6}{5}xc = \frac{4}{5}xc$. He saves $\frac{\frac{1}{5}xc}{xc} = \text{(A)}20\%$.

Because they do not give you a given amount of distance, we'll just make that distance $3x$ miles. Then, we find that the new car will use $2 * 1.2 = 2.4x$. The old car will use $3x$. Thus the answer is $\frac{3 - 2.4}{3} = \frac{0.6}{3} = \frac{20}{100} = \text{(A)}20\%$.

You can find that the ratio of fuel used by the old car and the new car in a same amount of distance is $3 : 2$, and the ratio between the fuel price of these two cars is $5 : 6$. Therefore, by multiplying these two ratios, we get that the costs of using these two cars is $15 : 12 = 5 : 4$, So the percentage of money saved is $1 - \frac{5}{4} = \text{(A)}20\%$.

- 13 There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes? (2017 AMC 10B Problem, Question#13)

- A. 1 B. 2 C. 3 D. 4 E. 5

Answer C

Solution By PIE (Property of Inclusion/Exclusion), the answer is $10 + 13 + 9 - 9 - 20 = 3$.

- 14 An integer N is selected at random in the range $1 \leq N \leq 2020$. What is the probability that the remainder when N^{16} is divided by 5 is 1? (2017 AMC 10B Problem, Question#14)
- A. $\frac{1}{5}$ B. $\frac{2}{5}$ C. $\frac{3}{5}$ D. $\frac{4}{5}$ E. 1

Answer D

Solution By Fermat's Little Theorem, $N^{16} = (N^4)^4 \equiv 1 \pmod{5}$, when N is relatively prime to 5. Hence, this happens with probability $\frac{4}{5}$.

Note that the patterns for the units digits repeat, so in a sense we only need to find the patterns for the digits 0 – 9. The pattern for 0 is 0, no matter what power, so 0 doesn't work. Likewise, the pattern for 5 is always 5. Doing the same for the rest of the digits, we find that the units digits of $1^{16}, 2^{16}, 3^{16}, 4^{16}, 6^{16}, 7^{16}, 8^{16}$ and 9^{16} all have the remainder of 1 when divided by 5, so $\frac{4}{5}$.

We can use modular arithmetic for each residue of $n \pmod{5}$.

If $n \equiv 0 \pmod{5}$, then $n^{16} \equiv 0^{16} \equiv 0 \pmod{5}$,

If $n \equiv 1 \pmod{5}$, then $n^{16} \equiv 1^{16} \equiv 1 \pmod{5}$,

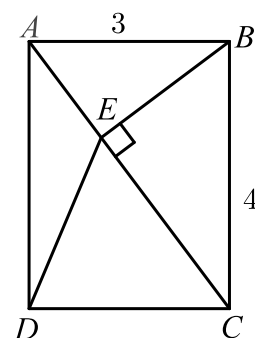
If $n \equiv 2 \pmod{5}$, then $n^{16} \equiv (n^2)^8 \equiv (2^2)^8 \equiv 4^8 \equiv (-1)^8 \equiv 1 \pmod{5}$.

If $n \equiv 3 \pmod{5}$, then $n^{16} \equiv (n^2)^8 \equiv (3^2)^8 \equiv 9^8 \equiv (-1)^8 \equiv 1 \pmod{5}$.

If $n \equiv 4 \pmod{5}$, then $n^{16} \equiv 4^{16} \equiv (-1)^{16} \equiv 1 \pmod{5}$.

In 4 out of the 5 cases, the result was $1 \pmod{5}$, and since each case occurs equally as $2020 \equiv 0 \pmod{5}$, the answer is $\frac{4}{5}$.

- 15 Rectangle $ABCD$ has $AB = 3$ and $BC = 4$. Point E is the foot of the perpendicular from B to diagonal $\overline{AC}$. What is the area of $\triangle AED$? (2017 AMC 10B Problem, Question#15)



- A. 1 B. $\frac{42}{25}$ C. $\frac{28}{15}$ D. 2 E. $\frac{54}{25}$

Answer E

Solution First, note that $AC = 5$ because ABC is a right triangle. In addition, we have

$AB \cdot BC = 2[ABC] = AC \cdot BE$, so $BE = \frac{12}{5}$. Using similar triangles within ABC , we get that $AE = \frac{9}{5}$ and $CE = \frac{16}{5}$. Let F be the foot of the perpendicular from E to AB .

Since EF and BC are parallel, $\triangle AFE$ is similar to $\triangle ABC$. Therefore, we have

$\frac{AF}{AB} = \frac{AE}{AC} = \frac{9}{25}$. Since $AB = 3$, $AF = \frac{27}{25}$. Note that AF is an altitude of $\triangle AED$ from AD , which has length 4. Therefore, the area of $\triangle AED$ is $\frac{1}{2} \cdot \frac{27}{25} \cdot 4 = \frac{54}{25}$.

Alternatively, we can use coordinates. Denote D as the origin. We find the equation for AC as $y = -\frac{4}{3}x + 4$, and BE as $y = \frac{3}{4}x + \frac{7}{4}$. Solving for x yields $\frac{27}{25}$. Our final answer then becomes $\frac{1}{2} \cdot \frac{27}{25} \cdot 4 = \frac{54}{25}$.

We note that the area of ABE must equal area of AED because they share the base and the height of both is the altitude of congruent triangles. Therefore, we find the area of ABE to be $\frac{1}{2} \cdot \frac{9}{5} \cdot \frac{12}{5} = \frac{54}{25}$.

We know all right triangles are $5-4-3$, so the areas are proportional to the square of like sides. Area of ABE is $\left(\frac{3}{5}\right)^2$ of $ABC = \frac{54}{25}$. Using similar logic in Solution 3, Area of AED is the same as ABE .

16 How many of the base-ten numerals for the positive integers less than or equal to 2017 contain the digit 0? (2017 AMC 10B Problem, Question#16)

- A. 469 B. 471 C. 475 D. 478 E. 481

Answer A

Solution We can use complementary counting. There are **2017** positive integers in total to consider, and there are **9** one-digit integers, $9 \cdot 9 = 81$ two digit integers without a zero, $9 \cdot 9 \cdot 9$ three digit integers without a zero, and $9 \cdot 9 \cdot 9 = 729$ four-digit integers starting with a 1 without a zero. Therefore, the answer is

$$2017 - 9 - 81 - 729 - 729 = 469.$$

- 17** Call a positive integer monotonous if it is a one-digit number or its digits, when read from left to right, form either a strictly increasing or a strictly decreasing sequence. For example, **3,23578**, and **987620** are monotonous, but **88,7434**, and **23557** are not. How many monotonous positive integers are there? (2017 AMC 10B Problem, Question#17)
- A. 1024 B. 1524 C. 1533 D. 1536 E. 2048

Answer B

Solution Case 1: monotonous numbers with digits in ascending order.

There are $\sum_{n=1}^9 \binom{9}{n}$ ways to choose n digits from the digits **1** to **9**. For each of these ways, we can generate exactly one monotonous number by ordering the chosen digits in ascending order. Note that **0** is not included since it will always be a leading digit and that is not allowed. Also, $\emptyset$ (the empty set) isn't included because it doesn't generate a number. The sum is equivalent to

$$\sum_{n=1}^9 \binom{9}{n} - \binom{9}{0} = 2^9 - 1 = 511.$$

Case 2: monotonous numbers with digits in descending order.

There are $\sum_{n=0}^{10} \binom{10}{n}$ ways to choose n digits from the digits **0** to **9**. For each of these ways, we can generate exactly one monotonous number by ordering the chosen digits in descending order. Note that **0** is included since we are allowed to end numbers with zeros. However, $\emptyset$ (the empty set) still isn't included because it doesn't generate a number. The sum is equivalent to

$$\sum_{n=0}^{10} \binom{10}{n} - \binom{10}{0} = 2^{10} - 1 = 1023.$$

We discard the number 0 since it is not positive. Thus there are **1022** here.

Since the 1-digit numbers 1 to 9 satisfy both case 1 and case 2, we have overcounted by 9. Thus there are $511 + 1022 - 9 = 1524$ monotonous numbers.

Like Solution 1, divide the problem into an increasing and decreasing case:

Case 1: Monotonous numbers with digits in ascending order.

Arrange the digits 1 through 9 in increasing order, and exclude 0 because a positive integer cannot begin with 0.

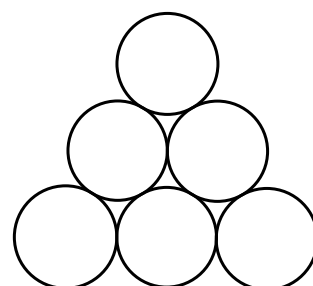
To get a monotonous number, we can either include or exclude each of the remaining 9 digits, and there are $2^9 = 512$ ways to do this. However, we cannot exclude every digit at once, so we subtract 1 to get $512 - 1 = 511$ monotonous numbers for this case.

Case 2: Monotonous numbers with digits in descending order.

This time, we arrange all 10 digits in decreasing order and repeat the process to find $2^{10} = 1024$ ways to include or exclude each digit. We cannot exclude every digit at once, and we cannot include only 0, so we subtract 2 to get $1024 - 2 = 1022$ monotonous numbers for this case. At this point, we have counted all of the single-digit monotonous numbers twice, so we must subtract 9 from our total.

Thus our final answer is $511 + 1022 - 9 = 1524$.

- 18** In the figure below, 3 of the 6 disks are to be painted blue, 2 are to be painted red, and 1 is to be painted green. Two paintings that can be obtained from one another by a rotation or a reflection of the entire figure are considered the same. How many different paintings are possible? (2017 AMC 10B Problem, Question#18)



A. 6

B. 8

C. 9

D. 12

E. 15

Answer D

Solution First we figure out the number of ways to put the blue 3 disks. Denote the spots to put the disks as 1 – 6 from left to right, top to bottom. The cases to put the blue disks are $(1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 2, 6), (2, 3, 5), (1, 4, 6)$. For each of those cases we can easily figure out the number of ways for each case, so the total amount is $2 + 2 + 3 + 3 + 1 + 1 = 12$.

19 Let $\triangle ABC$ be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$? (2017 AMC 10B Problem, Question#19)

A. 9 : 1

B. 16 : 1

C. 25 : 1

D. 36 : 1

E. 37 : 1

Answer E

Solution Solution 1

Note that by symmetry, $\triangle A'B'C'$ is also equilateral. Therefore, we only need to find one of the sides of $\triangle A'B'C'$ to determine the area ratio. WLOG, let $AB = BC = CA = 1$. Therefore, $BB' = 3$ and $BC' = 4$. Also, $\angle B'BC' = 120^\circ$, so by the Law of Cosines, $B'C' = \sqrt{37}$. Therefore, the answer is $(\sqrt{37})^2 : 1^2 = 37 : 1$.

Solution 2

As mentioned in the first solution, $\triangle A'B'C'$ is equilateral. WLOG, let $AB = 2$. Let D be on the line passing through AB such that $A'D$ is perpendicular to AB . Note that $\triangle A'DA$ is a $30 - 60 - 90$ with right angle at D . Since $AA' = 6$, $AD = 3$ and $A'D = 3\sqrt{3}$. So we know that $db' = 11$. Note that $\triangle A'DB'$ is a right triangle with right angle at D . So by the Pythagorean theorem, we find $A'B' = \sqrt{(3\sqrt{3})^2 + 11^2} = 2\sqrt{37}$. Therefore, the answer is $(2\sqrt{37})^2 : 2^2 = 37 : 1$.

20 The number $21! = 51,090,942,171,709,440,000$, has over 60,000 positive integer divisors.

One of them is chosen at random. What is the probability that it is odd? (2017 AMC 10B Problem, Question#20)

- A. $\frac{1}{21}$ B. $\frac{1}{19}$ C. $\frac{1}{18}$ D. $\frac{1}{2}$ E. $\frac{11}{21}$

Answer B

Solution Solution 1

We note that the only thing that affects the parity of the factor are the powers of 2.

There are $10 + 5 + 2 + 1 = 18$ factors of 2 in the number. Thus, there are 18 cases in which a factor of $21!$ would be even (have a factor of 2 in its prime factorization), and 1 case in which a factor of $21!$ would be odd. Therefore, the answer is $\frac{1}{19}$.

Solution 2

Consider how to construct any divisor D of $21!$. First by Legendre's theorem for the divisors of a factorial, we have that there are a total of 18 factors of 2 in the number. D can take up either 0,1,2,3,..., or all 18 factors of 2, for a total of 19 possible cases. In order for D to be odd, however, it must have 0 factors of 2, meaning that there is a probability of 1 case/19 cases = $\frac{1}{19}$.

21 In $\triangle ABC$, $AB = 6$, $AC = 8$, $BC = 10$, and D is the midpoint of $\overline{BC}$. What is the sum of the radii of the circles inscribed in $\triangle ADB$ and $\triangle ADC$? (2017 AMC 10B Problem, Question#21)

- A. $\sqrt{5}$ B. $\frac{11}{4}$ C. $2\sqrt{2}$ D. $\frac{17}{6}$ E. 3

Answer D

Solution We note that by the converse of the Pythagorean Theorem, $\triangle ABC$ is a right triangle with a right angle at A . Therefore, $AD = BD = CD = 5$, and $[ADB] = [ADC] = 12$. Since

$A = rs$, the inradius of $\triangle ADB$ is $\frac{12}{5+5+6} = \frac{3}{2}$, and the inradius of $\triangle ADC$ is

$$\frac{12}{5+5+8} = \frac{4}{3}. \text{ Adding the two together, we have } \frac{17}{6}.$$

We have $\triangle ABC$ a right triangle by dividing each side lengths by 2 to create a well known 3 – 4 – 5 triangle. We also can know that the median of a right triangle must be equal to half the hypotenuse. Using this property, we have $BD = 5, CD = 5$, and $AD = 5$. Now, we can use the Heron's formula to get the area of $\triangle ABD$ as $\sqrt{8(2)(3)(3)} = \sqrt{144} = 12$. Afterward, we can apply this formula again on $\triangle ADC$ to get the area as $\sqrt{9(4)(4)(1)} = \sqrt{144} = 12$. Notice we want the inradius. We can use another property, which is $A = rs$. This states that $Area = Radius(Semiperimeter)$. (This can be proved by connecting the center of the inscribed circles to the vertices and we can notice the inradius is just the heights of each of the three triangles divided) Finally, we can derive the radii of each inscribed circle. Plugging the semiperimeter and area into the formula, we have $12 = 8r$ and $12 = 9r$ for $\triangle ABD$ and $\triangle ADC$, respectively. Simplifying, we have the radii lengths as $r = \frac{3}{2}$ and $\frac{4}{3}$. We want the sum, so we have $\frac{17}{6}$, or (D).

- 22 The diameter AB of a circle of radius 2 is extended to a point D outside the circle so that $BD = 3$. Point E is chosen so that $ED = 5$ and line ED is perpendicular to line AD . Segment AE intersects the circle at a point C between A and E . What is the area of $\triangle ABC$? (2017 AMC 10B Problem, Question#22)

- A. $\frac{120}{37}$ B. $\frac{140}{39}$ C. $\frac{145}{39}$ D. $\frac{140}{37}$ E. $\frac{120}{31}$

Answer D

Solution Solution 1

Notice that ADE and ABC are right triangles.

Then $AE = \sqrt{7^2 + 5^2} = \sqrt{74}$, $\sin DAE = \sin ABC = \frac{5}{\sqrt{74}} = \sin BAE = \frac{BC}{4}$, so $BC = \frac{20}{\sqrt{74}}$.

We also find that $AC = \frac{28}{\sqrt{74}}$, and thus the area of ABC is $\frac{\frac{20}{\sqrt{74}} \cdot \frac{28}{\sqrt{74}}}{2} = \frac{560}{74} = \frac{140}{37}$.

Solution 2

We note that $\triangle ACB \sim \triangle ADE$ by AA similarity. Also, since the area of

$\triangle ADE = \frac{7 \cdot 5}{2} = \frac{35}{2}$ and $AE = \sqrt{74}$, $\frac{[ABC]}{[ADE]} = \left(\frac{AC}{AE}\right)^2 = \left(\frac{4}{\sqrt{74}}\right)^2$, so the area of

$$\triangle ABC = \frac{140}{37}.$$

Solution 3

As stated before, note that $\triangle ACB, \triangle ADE$. By similarity, we note that $\frac{AC}{BC}$ is

equivalent to $\frac{7}{5}$. We set AC to $7x$ and BC to $5x$. By the Pythagorean Theorem,

$(7x)^2 + (5x)^2 = 4^2$. Combining, $49x^2 + 25x^2 = 16$. We can add and divide to get

$x^2 = \frac{8}{37}$. We square root and rearrange to get $x = \frac{2\sqrt{74}}{37}$. We know that the legs of

the triangle are $7x$ and $5x$. Multiplying x by 7 and 5 eventually gives us $\frac{14\sqrt{74}}{37}$ and $\frac{10\sqrt{74}}{37}$.

We divide this by 2, since $\frac{1}{2}bh$ is the formula for a triangle. This gives us $\frac{140}{37}$.

Solution 4

Let's call the center of the circle that segment AB is the diameter of, O . Note that

$\triangle ODE$ is an isosceles right triangle. Solving for side OE , using the Pythagorean

theorem, we find it to be $5\sqrt{2}$. Calling the point where segment OE intersects circle

O , the point I , segment IE would be $5\sqrt{2} - 2$. Also, noting that $\triangle ADE$ is a right

triangle, we solve for side AE , using the Pythagorean Theorem, and get $\sqrt{74}$. Using

Power of Point on point E , we can solve for CE . We can subtract CE from AE to find

AC and then solve for CB using Pythagorean theorem once more $(AE)(CE) =$

$$(\text{Diameter of circle } O + IE)(IE) \rightarrow \sqrt{74}(CE) = (5\sqrt{2} + 2)(5\sqrt{2} - 2) \Rightarrow CE = \frac{23\sqrt{74}}{37},$$

$$AC = AE - CE \rightarrow AC = \sqrt{74} - \frac{23\sqrt{74}}{37} \Rightarrow AC = \frac{14\sqrt{74}}{37}, \text{ Now to solve for } CB:$$

$$AB^2 - AC^2 = CB^2 \rightarrow 4^2 + \frac{14\sqrt{74}^2}{37} = CB^2 \Rightarrow CB = \frac{10\sqrt{74}}{37},$$

Note that $\triangle ABC$ is a right triangle because the hypotenuse is the diameter of the

circle. Solving for area using the bases AC and BC , we get the area of triangle ABC

to be $\frac{140}{37}$.

23 Let $N = 123456789101112 \dots 4344$ be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45? (2017 AMC 10B Problem, Question#23)

- A. 1 B. 4 C. 9 D. 18 E. 44

Answer C

Solution We only need to find the remainders of N when divided by 5 and 9 to determine the answer. By inspection, $N \equiv 4 \pmod{5}$, The remainder when N is divided by 9 is $1 + 2 + 3 + 4 + \dots + 1 + 0 + 1 + 1 + 1 + 2 + \dots + 4 + 3 + 4 + 4$, but since $10 \equiv 1 \pmod{9}$, we can also write this as $1 + 2 + 3 + \dots + 10 + 11 + 12 + \dots + 43 + 44 = \frac{44 \cdot 45}{2} = 22 \cdot 45$, which has a remainder of 0 mod 9. Therefore, by inspection, the answer is (C) 9

Note: the sum of the digits of N is 270.

Noting the solution above, we try to find the sum of the digits to figure out its remainder when divided by 9. From 1 thru 9, the sum is 45, 10 thru 19, the sum is 55, 20 thru 29 is 65, and 30 thru 39 is 75. Thus the sum of the digits is

$45 + 55 + 65 + 75 + 4 + 5 + 6 + 7 + 8 = 240 + 30 = 270$, and thus N is divisible by 9.

Now refer to the above solution $N \equiv 4 \pmod{5}$ and $N \equiv 0 \pmod{9}$. From this information, we can conclude this information, we can conclude that $N \equiv 54 \pmod{45}$ and $N \equiv 54 \pmod{9}$.

Therefore $N \equiv 54 \pmod{45}$ and $N \equiv 9 \pmod{45}$ so the remainder is (C) 9.

Because a number is equivalent to the sum of its digits modulo 9, we have that

$N \equiv 1 + 2 + 3 + 4 + 5 + \dots + 44 \equiv \frac{44 \cdot 45}{2} \equiv 0 \pmod{9}$. Furthermore, we see that $N - 9$ ends in the digit 5 and thus is divisible by 5, so $N - 9$ is divisible by 45, meaning the remainder is (C) 9.

24 The vertices of an equilateral triangle lie on the hyperbola $xy = 1$, and a vertex of this hyperbola is the centroid of the triangle. What is the square of the area of the triangle? (2017 AMC 10B Problem, Question#24)

- A. 48 B. 60 C. 108 D. 120 E. 169

Answer C

Solution WLOG, let the centroid of $\triangle ABC$ be $I = (-1, -1)$. The centroid of an equilateral triangle is the same as the circumcenter. It follows that the circumcircle must intersect the graph exactly three times.

Therefore, $A = (1, 1)$, so $AI = BI = CI = 2\sqrt{2}$, so since $\triangle AIB$ is isosceles and $\angle AIB = 120^\circ$, then by Law of Cosines, $AB = 2\sqrt{6}$. Alternatively, we can use the fact that the circumradius of an equilateral triangle is equal to $\frac{8}{\sqrt{3}}$. Therefore, the area of the triangle is $\frac{(2\sqrt{6})^2 \sqrt{3}}{4} = 6\sqrt{3}$, so the square of the area of the triangle is (C)108.

WLOG, let the centroid of $\triangle ABC$ be $G = (-1, -1)$. Then, one of the vertices must be the other curve of the hyperbola. WLOG, let $A = (1, 1)$. Then, point B must be the reflection of C across the

line $y = x$, so let $B = \left(a, \frac{1}{a}\right)$, and $C = \left(\frac{1}{a}, a\right)$,

where $a < -1$. Because G is the centroid, the average of the x -coordinates of the vertices of the triangle is -1 . So we know that $a + \frac{1}{a} + 1 = -3$, Multiplying by a and solving gives us $a = -2 - \sqrt{3}$.

So $B = (-2 - \sqrt{3}, -2 + \sqrt{3})$, and $C = (-2 + \sqrt{3}, -2 - \sqrt{3})$. So $BC = 2\sqrt{6}$, and finding the square of the area gives us (C)108.

WLOG, let the centroid of $\triangle ABC$ be $G = (1, 1)$ and let point A be $(-1, -1)$. It is known that the centroid is equidistant from the three vertices of $\triangle ABC$. Because we have the coordinates of both A and G, we know that the distance from G to any vertex of $\triangle ABC$ is $\sqrt{(1 - (-1))^2 + (1 - (-1))^2} = 2\sqrt{2}$. Therefore, $AG = BG = CG = 2\sqrt{2}$. It follows that from $\triangle ABG$, where $AG = BG = 2\sqrt{2}$ and

$\angle AGB = \frac{360^\circ}{3} = 120^\circ$ $[\triangle ABG] = \frac{(2\sqrt{2})^2 \cdot \sin(120)}{2} = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$ using the formula for the area of a triangle with sine $\left([\triangle ABC] = \frac{1}{2}AB \cdot BC \sin(\angle ABC)\right)$.

Because $\triangle ACG$ and $\triangle BCG$ are congruent to $\triangle ABG$, they also have an area of $2\sqrt{3}$.

Therefore $[\triangle ABC] = 3(2\sqrt{3}) = 6\sqrt{3}$. Squaring that gives us the answer of (C) 108.

WLOG, let the centroid of the triangle be $(1, 1)$. By symmetry, the other vertex is $(-1, -1)$. The distance between these two points is $2\sqrt{2}$, so the height of the triangle is $3\sqrt{2}$, the side length is $2\sqrt{6}$, and the area is $6\sqrt{3}$, yielding an answer of (C) 108.

25 Last year Isabella took 7 math tests and received 7 different scores, each an integer between 91 and 100, inclusive. After each test she noticed that the average of her test scores was an integer. Her score on the seventh test was 95. What was her score on the sixth test? (2017 AMC 10B Problem, Question#25)

- A. 92 B. 94 C. 96 D. 98 E. 100

Answer E

Solution Since all of the scores are from 91 – 100, we can 'subtract' 90 off from all of the scores. Basically, we're looking at the units digits except for 100; we're looking at 10 in this case. Since the last score was a 95, the sum of the scores from the first six tests must be $2 \pmod{7}$ and $0 \pmod{6}$. Trying out a few cases, the only solution possible is 30 (this is from adding numbers 1 – 10). The sixth test score must be $0 \pmod{5}$ because $30 \equiv 0 \pmod{5}$. The only possible test scores are 95 and 100, so the answer is (E) 100.